Foundations of Cities*

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Abstract: How do people choose work and residence locations when commuting is costly and productivity spillovers, increasing returns to scale, or first nature advantage, reward the concentration of employment? We describe such an equilibrium city in a simple geography populated by agents with heterogenous preferences over workplace-residence pairs. The behavior of equilibrium cities is more complex than previously understood. Increasing returns to scale and productivity spillovers can disperse employment in empirically relevant parts of the parameter space. Heterogeneous preferences concentrate employment and residence. These results have important implications for the study of agglomeration economies and the specification of quantitative spatial models.

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1 Introduction

How do people arrange themselves when they are free to choose work and residence locations, when commuting is costly, and when some economic mechanism rewards the concentration of employment? This is the fundamental question of urban economics. To address it, we must specify the mechanism that rewards the concentration of employment and how strongly it operates. We consider three such mechanisms; first nature productivity advantages, local increasing returns to scale (IRS), and productivity spillovers. We characterize the resulting equilibrium city in a simple geography populated by agents with heterogenous preferences over workplace-residence pairs. We find that the equilibrium configuration of the city and spatial variation in rents and wages are sensitive to these foundations in surprising ways.

First nature productivity advantages, IRS, and productivity spillovers are all widely regarded as "agglomeration forces", i.e., economic mechanisms that lead to the agglomeration of economic activity. Because first nature advantage confers an advantage on particular places, and productivity spillovers and returns to scale confer productivity advantages on agglomerations of employment, it is assumed that they must lead to such agglomerations. We demonstrate that this relationship holds only for first nature productivity advantages. The relationship between IRS, spillovers, and the agglomeration of residence and employment is more complicated.

Equilibrium requires that profits be zero in all locations. An increase in returns to scale upsets this equilibrium condition as productivity increases more rapidly in more densely populated places. An increase in wages or rents in the more productive place can drive down profits to restore equilibrium. Alternatively, a shift of employment *away* from the more productive place means that larger returns to scale operates on locations whose differences in employment are smaller. This decreases productivity differences across locations and can also restore equilibrium. That is, in equilibrium IRS can be a force for dispersion, not for agglomeration. Our results demonstrate that this logic is not purely hypothetical. It operates in an empirically plausible part of the parameter space.

To understand the effects of productivity spillovers on the agglomeration of employment, consider an equilibrium in which location A has an advantage in productivity relative to other locations. In equilibrium, employment concentrates and wages and rents are correspondingly higher in location A. As productivity spillovers increase, production that occurs near location A also benefits from the higher productivity in A, but without the same high wages and rents. Thus, as productivity spillovers increase, production disperses from location A. That is, productivity spillovers can operate as a force for dispersion not agglomeration. We establish that logic governs equilibrium in the empirically relevant part of the parameter space when both spillovers and IRS are small.

Our model is organized around agents with heterogenous preferences over pairs of workplace and residence locations. Although preferences are not usually regarded as an agglomeration force, we find that preference heterogeneity leads to an 'average preference for central employment and residence'. The intuition behind such average preferences can be seen in a Ricardian or Armington trade model. When a location must trade with every other location, the central location has an advantage as the place where average freight costs are lowest. In our more complicated framework, average preferences for centrality arise because an average household commutes everywhere with positive probability.

Multiple equilibria are pervasive in our framework when IRS operates. We consider two notions of stability as possible equilibrium refinements. The first is an 'iterative stability condition' and the second is a novel notion of stability loosely related to trembling hand perfection. It is well known that a conventional stability condition must hold in a neighborhood of a fixed point if numerical methods are to converge to the fixed point.[1] We show that this iterative stability condition is not robust to alternative formulations of the equilibrium conditions with the same solutions. This implies that fixed-point algorithms cannot generally be relied on to find all possible equilibria in models with multiple equilibria. We avoid this problem by solving our model analytically. This, in part, motivates our use of a highly stylized geography. This problem also motivates our definition of a novel stability condition. This notion of stability is similar to game theoretic notions of stability, like trembling hand perfection. It has two important advantages. The first is conceptual, like our model, our notion of stability is static. Second, it is tractable, and we provide an algorithm that allows us to determine whether any particular equilibrium is stable. This allows us to show that the corner equilibria

[1]For fixed point algorithms to converge, it is necessary that the Jacobian of the objective have eigenvalues with absolute value less than one at the fixed point.

that are pervasive under IRS are unstable, along with any equilibria that are 'near enough' to these corner equilibria.

These results are important for three reasons. First, cities are among the most important economic phenomena of the modern world. More than half the world's population lives in them and this share is rising rapidly. Understanding why cities are organized as they are is an economic problem of the first order. While there is an important existing literature on this question, it applies to a subset of the cases we consider. Existing results impose exogenous restrictions on location choices; are restricted to cases where the rewards for concentration are 'small enough' that multiple equilibria do not arise; restrict attention to technologies where only spillovers operate; or consider specific numerical examples. In contrast, we provide a complete characterization of equilibrium in much of the parameter space and allow productivity spillovers, IRS, and first nature advantage.

Second, there is a large literature investigating the magnitude of 'agglomeration economies' by estimating how wages vary with employment density. Our results will suggest that the interpretation of such empirical relationships is subtle: in the absence of returns to scale in production, preference heterogeneity leads to a negative relationship between wages and density, even as it leads to increasing central city density; for sufficiently high levels of increasing returns to scale, the wage premium for density begins to decrease; productivity spillovers across locations may have opposite effects from increasing returns to scale. Thus, learning about the foundations of agglomeration, that is, the foundations of cities, from common research designs may be more difficult than has previously been understood.

Third, our model has many features in common with models used in the literature on quantitative spatial models (QSM). Given this, much of the intuition that governs equilibrium behavior in our model should be relevant to the empirically based frameworks on which the QSM literature is based. In particular, that IRS and productivity spillovers have different implications, the presence of an average preference for central work and residence, and the pervasiveness of multiple equilibria, should all inform efforts to formulate quantitative spatial models that strive to accurately represent real economies.

2 Literature

We can usefully partition the literature into an older urban economics literature and a more recent literature on QSM. Most papers in classical urban economics assume that households are homogenous, or that there are a small number of types; space is continuous and uniform, whether on a line or in a plane; and equilibrium cities are symmetric around a single exogenously selected point. The most influential model in this literature is the monocentric city model. This workhorse model rests on the assumption that the location of work is fixed exogenously at the center and households choose only their location of residence, although the model is otherwise quite general (Fujita, 1989).

The scarcity of papers describing conditions under which technological spillovers lead to spatial concentration of production testifies to the difficulty of characterizing equilibrium when both firms and households choose their locations. The first general statement of this problem appears in a pair of papers, Ogawa and Fujita (1980) and Fujita and Ogawa (1982). These landmark papers consider a simple setting where firms choose only their location and households choose only their places of work and residence, but firm productivity in each location benefits from spillovers from every location, with distant spillovers less beneficial than those nearby. The two papers differ in two important ways. First, productivity spillovers decay linearly with distance in Ogawa and Fujita (1980) and exponentially in Fujita and Ogawa (1982). Second, Ogawa and Fujita (1980) provides an analytical description of equilibrium, while Fujita and Ogawa (1982) provides only numerical results.

Lucas and Rossi-Hansberg (2002) revisit Fujita and Ogawa (1982) with two main changes. First, they allow firms and households to substitute between consumption and land. Second, they generalize the Fujita and Ogawa's description of spillovers in order to decompose it into pure spillovers and global increasing returns. They establish general existence and uniqueness results, but otherwise rely on numerical methods. They restrict attention to 'weak enough' global increasing returns and do not consider comparative statics around this parameter at all. Rather, they develop numerical comparative statics around changes in their pure spillovers measure. In a related work, Berliant *et al.* (2002) consider a city where the technology is a Cobb-Douglas function of land, labor and capital and spillovers depend on the average capital investment multiplied and a proxy for the spatial distribution of firms. They show that a monocentric city emerges under conditions similar to those in Ogawa and Fujita (1980) and Lucas and Rossi-Hansberg (2002).

These papers differ in the details of how they describe productivity spillovers, but all lead to the same basic conclusion. Productivity spillovers create an agglomeration force and land scarcity acts as a dispersion force. As the magnitude of spillovers increases relative to the cost of commuting, a monocentric city eventually emerges as the equilibrium outcome.

We extend this literature in three main ways. First, the classical urban economics literature considers only productivity spillovers, not IRS or first nature advantages. Second, we provide a more complete characterization of equilibrium. Third, we consider heterogeneous agents. Because our specification of heterogeneity nests 'no heterogeneity' as a limit case, this is a strict generalization of this older literature.

Quantitative spatial models (QSM) have recently been brought to bear on problems of urban economics (see Redding and Rossi-Hansberg (2017) for a survey). In this literature, cities consist of discrete sets of locations rather than continuous spaces, and are described by realistic rather than stylized geographies. In addition, this literature considers heterogenous households rather than the homogenous agents of classical urban economics. Finally, where the older literature tends to focus on analytic solutions and qualitative results, QSM focuses on the numerical evaluation of particular comparative statics in models that describe particular real world locations.

The QSM literature draws on a long history of scholarship on discrete choice models. Space is discrete and is described a matrix of pairwise commuting costs. These matrices are typically constructed to describe commuting costs between pairs of neighborhoods in the empirical application of interest. Households have heterogenous preferences over work-residence pairs and each household selects a unique pair. Locations are heterogenous in their amenities and productivity, and the possibility of endogenous agglomeration economies is sometimes considered. These features all appear in our model.

In an important series of papers, Allen and Arkolakis (with coauthors) study the existence and uniqueness of equilibrium in spatial models similar to ours. Briefly, Allen and Arkolakis (2014) considers a model with a reduced form description of the land market. There is no land market in the model by Allen, Arkolakis and Takahashi (2020). Allen, Arkolakis and Li (2020) set the shares of residential and commercial land in each neighborhood exogenously. Allen, Arkolakis and Li (2015) is difficult to compare to our model because they "consider a general firm technology (e.g. it could be constant or decreasing returns to scale)" (p.4), which seems to rule out IRS (see their equation (10)). While some of the Arkolakis and Allen results can probably be adapted to our setting, we do not investigate this possibility because our existence proof is straightforward and because our focus is on a characterization of multiple equilibria, an issue that the Arkolakis and Allen theorems do not address.

We contribute to this second literature in a number of ways. First, we provide a complete characterization of equilibrium throughout much of the parameter space. The existing literature restricts attention to parts of the parameter space where equilibrium is unique.[2] We characterize equilibrium for arbitrary levels of IRS. Second, we provide a number of new results. We establish the pervasive and surprising existence of corner equilibria. We establish that IRS can sometimes act to disperse employment rather than concentrate it. We establish that, in parts of the parameter space, increases in spillovers and IRS have opposite effects on the equilibrium configuration of employment. Finally, we establish that heterogeneity of preferences over workplace-residence location pairs gives rise to an average preference for central work and residence.

3 Model, equilibrium, and solution method

A *Model*

A city consists of three locations $i,j = -1,0,1$. Each location is endowed with one unit of land. This geography is the simplest with which to examine when activities concentrate in a land scarce center or disperse to a land abundant periphery. Simultaneously, it is also rich enough to exhibit novel and surprising behavior, and to refine our intuition about how economic forces operate to form equilibrium cities. As is always the case with models that have a small number of locations (e.g., Krugman (1991) or most trade models), our setting precludes immediate empirical application, and may rule out yet more complex phenomena.

[2]Allen and Donaldson (2022) is a partial exception that also studies parts of the parameter space 'close' to the region where equilibria are unique.

Against this, our approach has the advantages of tractability and transparency, and much of the intuition that we derive appears to be general.

The city is populated by a continuum [0,1] of households and by a competitive production sector whose size is endogenous. All households choose a residence i, a workplace j, their consumption of housing, and a tradable numeraire good. Households have heterogenous preferences over workplace-residence pairs, and household types parameterize preferences. Each household $\nu \in [0,1]$ has a type $\mathbf{z}(\nu) \equiv (z_{ij}(\nu)) \in \mathbb{R}_+^{3\times 3}$, a vector of non-negative real numbers, one for each possible workplace-residence pair ij. The mapping $\mathbf{z}(\nu) : [0,1] \to \mathbb{R}_+^{3\times 3}$ is such that the distribution of types is the product measure of 9 identical Fréchet distributions:

$$F(\mathbf{z}) \equiv \exp\left(-\sum_i \sum_j z_{ij}^{-\varepsilon}\right). \tag{1}$$

Thus, $\varepsilon \in (0,\infty)$ describes the heterogeneity of preferences. An increase in ε reduces preference heterogeneity and conversely.

Households commute between workplace and residence. Commuting from i to j involves an iceberg cost $\tau_{ij} \geq 1$. This cost is the same for all households and $\tau_{ij} = 1$ if and only if $i = j$. Commuting costs affect household utility directly.[3]

A household that lives at i and works at j has an indirect utility

$$V_{ij}(\nu) = z_{ij}(\nu) B_i D_j \frac{W_j}{\tau_{ij} R_i^\beta}, \tag{2}$$

where W_j is the wage paid at location j and R_i the land rent at i. In this expression, B_i stands for exogenous residential amenities at location i and D_j for exogenous amenities at the workplace j. Wages are the only source of income because land rent accrues to absentee landlords and constant returns to scale production guarantees that equilibrium profits are zero.

Using (1) – (2), the share s_{ij} of households who choose the location pair ij equals

$$s_{ij} = \frac{\left[B_i D_j W_j / \left(\tau_{ij} R_i^\beta\right)\right]^\varepsilon}{\sum_r \sum_s \left[B_r D_s W_s / \left(\tau_{rs} R_r^\beta\right)\right]^\varepsilon}, \tag{3}$$

[3]These preferences are widely used in quantitative models, e.g., Ahlfeldt *et al.* (2015), Monte *et al.* (2018) and Heblich *et al.* (2020) to mention a few.

where the equality stems from the Fréchet distribution assumption. All choices are simultaneous.

Equations (1) – (2) are well-known from the literature and apply to any geography. We now turn to the specific features of the linear city.

First, the iceberg commuting cost matrix is

$$\begin{pmatrix} \tau_{-1,-1} & \tau_{-1,0} & \tau_{-1,1} \\ \tau_{0,-1} & \tau_{0,0} & \tau_{0,1} \\ \tau_{1,-1} & \tau_{1,0} & \tau_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & \tau & \tau^2 \\ \tau & 1 & \tau \\ \tau^2 & \tau & 1 \end{pmatrix}, \tag{4}$$

where $\tau > 1$.

Second, the simple structure of the commuting cost matrix (4) facilitates introducing a measure of spatial frictions which encapsulates both the commuting cost τ and preference heterogeneity ε, i.e., the *spatial discount factor* defined by

$$\phi \equiv \tau^{-\varepsilon}. \tag{5}$$

The spatial discount factor $\phi \in (0,1)$ decreases with the level of commuting costs ($\tau \uparrow$) and increases with the heterogeneity of the population ($\varepsilon \downarrow$). Hence, ϕ may be high because either commuting costs are low, or the population is very heterogeneous, or both. It is easy to see that $\phi \to 1$ when $\tau \to 1$ or $\varepsilon \to 0$, while $\phi \to 0$ when $\tau \to \infty$ or $\varepsilon \to \infty$. This last relationship is of particular interest. When $\varepsilon \to \infty$ the heterogeneity of preferences disappears, so examining behavior as we approach this limit allows us to examine the implications of removing preference heterogeneity from the model.

Third, define a spatial pattern $\mathbf{X} = (X_{-1}, X_0, X_1)$ as a triple that specifies the values of variable X at each location i. Given our geography and to ease comparison with the urban economics literature, we focus on *symmetric* spatial patterns where $X_1 = X_{-1}$. We use lower case letters to indicate ratios of central to peripheral values in any given symmetric pattern, e.g., $b = B_0/B_1$, $d = D_0/D_1$, $w = W_0/W_1$, $r = R_0/R_1$.

Finally, define

$$\rho \equiv \left(br^{-\beta}\right)^{\varepsilon}, \quad \omega \equiv (dw)^{\epsilon}. \tag{6}$$

Recalling that b is the relative residential amenity and d the relative workplace amenity, ω is the *amenity-adjusted relative wage* and ρ the *inverse amenity-adjusted* *relative rent*. In the interests of brevity, we will generally refer to ρ as the *amenity-adjusted relative rent*. When context precludes confusion with r and w, we will also refer to ρ and ω simply as relative rents and wages.

Using symmetry and (4) – (6), the commuting flows (3) can be restated as

$$\begin{pmatrix} s_{11} & s_{10} & s_{1-1} \\ s_{01} & s_{00} & s_{0-1} \\ s_{-11} & s_{-10} & s_{-1-1} \end{pmatrix} = \frac{1}{\rho\omega + 2\phi(\rho+\omega) + 2(1+\phi^2)} \begin{pmatrix} 1 & \phi\omega & \phi^2 \\ \phi\rho & \rho\omega & \phi\rho \\ \phi^2 & \phi\omega & 1 \end{pmatrix}. \quad (7)$$

The appeal of equation (7) is that ρ and ω uniquely determine the commuting flows through very simple algebraic expressions.

Let M_i and L_j be the mass of residents and workers at $i,j = 0,1$:

$$\begin{aligned} M_0 &= s_{00} + 2s_{01}, \quad M_1 = s_{10} + (1+\phi^2)s_{11}, \\ L_0 &= s_{00} + 2s_{10}, \quad L_1 = s_{01} + (1+\phi^2)s_{11}. \end{aligned} \quad (8)$$

Equations (7) – (8) imply labor market clearing and population balance:

$$L_0 + 2L_1 = M_0 + 2M_1 = 1.$$

Let H_i be the amount of residential land and N_i the amount of commercial land at location i. Because each location i is endowed with one unit of land, land market[4] clearing also requires

$$H_i + N_i = 1. \quad (9)$$

Assume that the numeraire is produced under perfect competition and the production functions at locations $j = 0,1$ are, respectively,

$$Y_0 = A_0 L_0^{\alpha} N_0^{1-\alpha}, \qquad Y_1 = A_1 L_1^{\alpha} N_1^{1-\alpha}, \quad (10)$$

where A_j is location-specific TFP and $0 < \alpha < 1$. In line with the literature, we assume that increasing returns are localized while spillovers obey a negative exponential function across locations,

$$A_0 = C_0 L_0^{\gamma} + 2\delta L_1, \qquad A_1 = C_1 L_1^{\gamma} + \delta L_0 + \delta^2 L_1. \quad (11)$$

[4]To allow for corner equilibria, this condition should be written as $(H_i + N_i - 1)R_i = 0$.

Recalling our convention of denoting centrality ratios with lower case letters, we have $m = M_0/M_1$, $\ell = L_0/L_1$, $h = H_0/H_1$, $n = N_0/N_1$, $a = A_0/A_1$, and $c = C_0/C_1$. These are the ratios of central to peripheral quantities of residents, employment, residential land, commercial land, TFP, and first nature productivity.

Cost minimization implies that the relative demands for production factors are given by

$$\frac{W_j}{R_j} = \frac{\alpha}{1-\alpha}\frac{N_j}{L_j}. \tag{12}$$

Dividing the relative demand at $i = 0$ by the relative demand at $i = 1$, we get:

$$\frac{r}{w} = \frac{\ell}{n}. \tag{13}$$

To satisfy the zero profit condition, unit cost must equal the price of the numeraire, i.e.,

$$\frac{1}{A_i}\left(\frac{W_i}{\alpha}\right)^{\alpha}\left(\frac{R_i}{1-\alpha}\right)^{1-\alpha} = 1. \tag{14}$$

Dividing (14) at $i = 0$ by the corresponding condition at $i = 1$ yields

$$\frac{w^{\alpha}r^{1-\alpha}}{a(\ell)} = 1, \tag{15}$$

where $a(\ell)$ is the ratio of central and peripheral TFP,

$$a(\ell) \equiv \frac{A_0}{A_1} = \frac{C_0\left(\frac{\ell}{\ell+2}\right)^{\gamma} + \frac{2\delta}{\ell+2}}{C_1\left(\frac{1}{\ell+2}\right)^{\gamma} + \delta\frac{\ell+\delta}{\ell+2}}. \tag{16}$$

The expressions for TFP in (11) and relative TFP in (16) allow a parametric description of the three economic forces conventionally regarded as foundations for agglomerations of economic activity; first nature technical advantage, local increasing returns to scale, and spillovers. When $\gamma = \delta = 0$ and $c \neq 1$, then the center and periphery have different first nature advantages, but production is CRS and there are no spillovers. When $\delta = 0$, $c = 1$ and $\gamma > 0$, we have local IRS, but no spillovers. Finally, when $\gamma = 0$, $c = 1$ and $\delta > 0$, then spillovers affect productivity, but there is no local IRS or first nature advantage. Therefore, this description of TFP permits an investigation of how the organization of a city changes with the intensity and the mechanism that rewards the concentration of employment.

The expressions for TFP in (11) and relative TFP in (16) also describe the particular productivity processes on which much of the urban economics literature is based as special cases. For example, once we set $\delta = 0$, Ciccone and Hall (1996), Duranton and Puga (2004) and Allen and Arkolakis (2014) specify location-specific returns to scale just as in equation (16). On the other hand, if we set IRS to zero and adjust for our discrete geography, our definition of TFP mirrors Fujita and Ogawa (1982). In Fujita and Ogawa (1982), TFP at a location x in a linear city is given by

$$TFP(x) = \int_{-\infty}^{\infty} L(y) e^{-\sigma|x-y|} dy, \tag{17}$$

To see the correspondence between (17) and (11), let one unit of employment at one unit distance from x contribute $\delta = e^{-\sigma}$ to TFP, and at two units of distance, contributes $\delta^2 = e^{-2\sigma}$.[5] Notice that this suggests $\delta = 1$ as the upper bound of δ.

B ***Equilibrium***

We can now define a spatial equilibrium,

Definition 1 *A spatial equilibrium is a vector of spatial patterns for real quantities and prices such that: (i) all households make utility-maximizing choices of workplace, residence, housing, and consumption; (ii) the production sector minimizes cost in all locations; (iii) production sector makes zero profit in all locations; and (iv) all markets at each location clear.*

We now turn to a characterization of equilibrium. To begin, define

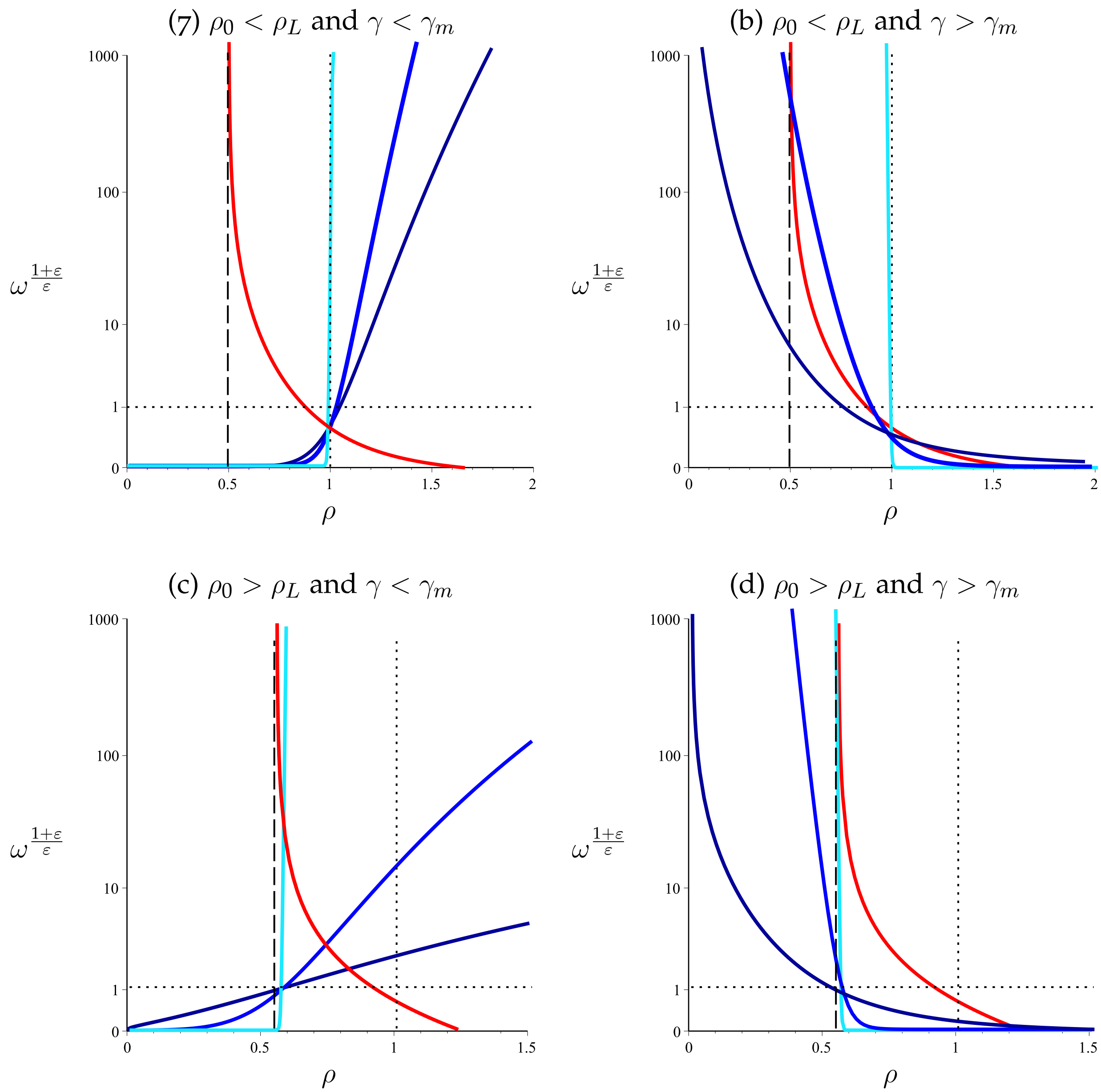
$$\eta \equiv \frac{\alpha\beta}{1-\alpha} > 0, \tag{18}$$

$$\psi \equiv \frac{(1-\alpha)(1+\varepsilon)}{\alpha\beta\varepsilon} = \frac{1+\varepsilon}{\eta\varepsilon}. \tag{19}$$

Proposition 1 shows that in equilibrium, employment, commercial land, and residential land patterns can all be expressed in terms of just ρ and ω.

[5] Lucas and Rossi-Hansberg (2002) and Ahlfeldt *et al.* (2015) use two very similar but slightly different versions of (17). This complicates comparisons of results.

Figure 1: Graphical demonstration of equilibrium for a range of parameter values.



Notes: These figures illustrate equilibrium in twelve different cases. In all panels, f is given by the red line. The blue lines describe g. In the left two panels, darker colors of blue indicate smaller values of γ and in the right two panels darker colors of blue indicate larger values of γ.

Proposition 1 *Spatial equilibrium is uniquely determined by ρ and ω. In particular,*

$$L_0 = \frac{\omega(\rho+2\phi)}{\omega(\rho+2\phi)+2(\phi\rho+1+\phi^2)}, \qquad L_1 = \frac{1-L_0}{2} \tag{20}$$

$$N_0 = \frac{\rho+2\phi}{\rho+2\phi+\eta\rho\left(1+2d\phi\omega^{-\frac{1+\epsilon}{\epsilon}}\right)}, \quad N_1 = \frac{1+\phi\rho+\phi^2}{1+\phi\rho+\phi^2+\eta\left(\frac{\phi\omega^{\frac{1+\epsilon}{\epsilon}}}{d}+1+\phi^2\right)} \tag{21}$$

$$H_0 = 1-N_0, \qquad H_1 = 1-N_1. \tag{22}$$

Proof: See Appendix A.

This proposition requires four comments. First, although Proposition 1 restricts attention to variables required for later derivations, all endogenous quantities in the economy can written in terms of ρ and ω. Second, the proof of Proposition 1 shows that the expressions for L (20) follow immediately from utility maximization, while the expressions for H and N, (21) and (22), also require market clearing. Third, the labor centrality ratio, ℓ, will play an important role in our analysis. Using Proposition 1, this ratio may be written

$$\ell(\rho,\omega) \equiv \frac{L_0}{L_1} = \omega\frac{\rho+2\phi}{\phi\rho+1+\phi^2}. \tag{23}$$

Finally, and as expected, equilibrium labor supply at the central location increases with the relative wage, w and decreases with the relative land rent, r.

C *Solution method*

We can use Proposition 1 to write the cost minimization and zero profit conditions (13) and (15) in terms of ρ and ω. Loosely, if we solve the two resulting equations for ω and equate them, we are left with a single equation in ρ that is sufficient to determine the interior equilibria. Proposition 2 provides the foundation for stating this result precisely.

Proposition 2 *Assume $\gamma \neq \gamma_m \equiv \alpha/\varepsilon$. Then, a pair (ρ^*,ω^*) is an interior equilibrium if and only if it solves the two equations:*

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho) \equiv d\frac{\phi b^{\frac{1}{\beta}}\rho - 2\eta\phi\rho^{1+\frac{1}{\beta\varepsilon}} + (1+\phi^2)(1+\eta)b^{\frac{1}{\beta}}}{(1+\eta)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - \eta\phi b^{\frac{1}{\beta}}}, \tag{24}$$

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} = \begin{cases} g(\rho;\gamma) \equiv \Phi^{\frac{1}{\alpha-\gamma\varepsilon}}\rho^{\frac{\alpha\psi}{\alpha-\gamma\varepsilon}}\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\right)^{\frac{\gamma\varepsilon}{\alpha-\gamma\varepsilon}\frac{1+\varepsilon}{\varepsilon}} & \textit{when } \delta = 0, \\ \left[\frac{1}{\Psi}a(\ell(\rho,\omega))\right]^{\frac{1+\varepsilon}{\alpha}}\rho^{\psi} & \textit{when } \delta > 0, \end{cases} \tag{25}$$

where $\Phi \equiv c^{\varepsilon\psi\eta} d^{\alpha\varepsilon\psi\eta} b^{-\alpha\varepsilon\psi}$ *and* $\Psi \equiv \left(b^{\frac{1}{\eta}} d^{-1}\right)^{\alpha}$.

Proof: See in Appendix B.

Equations (24) and (25) are complicated, but the intuition behind them is simple. The expression for f results from substituting L and N from Proposition 1 into equation (13). Equation (13) follows directly from cost minimization. The expression for L in Proposition 1 follows immediately from utility maximization, while the expressions for N_0, and N_1, (21) and (22), also require market clearing. Thus, f describes a locus of amenity adjusted relative prices, (ρ,ω), satisfying cost minimization, utility maximization, and land market clearing. Simplifying, we call f the market-clearing locus. Note that because f does not require the zero profit condition to hold, parameters that affect productivity directly, c, δ, and γ, do not appear in equation (24).

The expression for g, or more generally, equation (25), results from substituting (23) into (15). Equation (15) follows from cost minimization and the zero profit condition, and (23) directly from utility maximization. Thus, g describes a locus of relative prices satisfying cost minimization, utility maximization, and zero profits (but not land market clearing). Simplifying, we call g the zero-profit locus.

Recalling the definition of a spatial equilibrium, a pair of amenity adjusted relative prices (ρ,ω) that lies on the market clearing and zero profit curves is an equilibrium. When $\delta = 0$ and there are no productivity spillovers we can equate (24) and (25) to arrive at a single equation in ρ that determines the interior equilibria. In this case, we study the equilibrium behavior of our discrete linear city by studying the solution(s) of the equation,

$$f(\rho) = g(\rho; \gamma). \tag{26}$$

We show the existence of an interior equilibrium by showing that (26) has an interior solution. We determine the number of possible interior equilibria by determining the number of interior solutions of (26).

We cannot apply this solution method when $\delta > 0$. In this case, the left hand side of (25) is transcendental and no closed form expression for ω exists. The remainder of the section considers the case when $\delta = 0$. We postpone our treatment of the case when $\delta > 0$ to section 6. Note that when $\gamma = \gamma_m$, g is discontinuous, and so this case requires special attention.

Lemma 1 in Appendix C establishes that, as shown by the red line in all panels of figure 1, the market clearing locus, $f(\rho)$, is a positive, continuous function that declines monotonically from a positive asymptote at ρ_m, to zero at ρ_M. To develop intuition about market clearing locus, consider an increase in ω that reflects an increase in the central wage W_0. As W_0 increases, cost minimizing central firms substitute away from central labor towards land and utility maximizing central household spend more on residential land. If the central land market is to clear, R_0 must increase and, therefore, ρ decrease. This gives the required negative relationship between ω and ρ along f. Mechanically, ω goes to zero or infinity as W_0 or W_1 approaches zero. Lemma 1 also shows that ρ_m and ρ_M are the corresponding corner values of ρ along f. It follows that the relationship between ρ and ω must be negative along f for ρ between ρ_m and ρ_M, and that values of ρ outside this interval imply economically irrelevant negative wages.

The left two panels of figure 1, (7) and (c), describe g for three different values of $\gamma < \gamma_m$, with dark blue the smallest, light blue the largest, medium blue in between. The right two panels, (b) and (d), are the same as the left, but consider $\gamma > \gamma_m$. Here, the light blue line traces g for the smallest value of γ, dark blue uses the largest value, medium blue is intermediate value, and all three are greater than γ_m. In all panels, the light blue line describes g for a value of γ that is close to the singularity γ_m and darker colors are progressively further away.

Lemma 2 in Appendix D establishes three properties of g. For $\gamma < \gamma_m$, g is an increasing function that converges to an increasing step function as γ approaches γ_m from below. For $\gamma > \gamma_m$, g is a decreasing function that converges to a decreasing step function as γ approaches γ_m from above. Finally, the unique value of ρ at which the step occurs, ρ_L, is strictly between zero and one.

Our intuition about the behavior of the zero profit locus is based on the observation that as γ increases, there are three quantities that can adjust to preserve the zero profit condition given in (14), wages, rents, and employment (because $A_i = L_i^\gamma$). For small values of γ, we can ignore changes in A_i when we think about how the zero profit locus behaves.[6] When wages go up in a location, preserving the zero profit condition requires that rent must decline. This gives us a negative relationship between w and r, and thus, the positive relationship between ω and ρ that we see in the two left panels of figure 1.

[6]When $0 < L_i < 1$, for γ small, L_i is close to one unless L_i is close to zero.

As γ increases beyond γ_m, A_i becomes more sensitive to changes in γ, partly because L_i^γ becomes more sensitive to small adjustments in employment.[7] As a consequence, increases in γ lead mechanically to changes in the wage, and still larger changes of the same sign in A_i. Preserving the zero profit condition now requires that wages and rents must move in the same direction. Thus, we have a positive relationship between w and r, or the negative relationship between ω and ρ that we see in the right two panels of figure 2.

The singularity in g that arises when $\gamma = \gamma_m$ and $\rho = \rho_L$ arises as employment concentrates entirely in the center or periphery (on the zero profit locus). In this case, the relative zero profit condition (26) becomes invariant to changes in relative wages and thus creates the step in g.

Figure 1 permits a graphical solution of equation (26), and hence a description of equilibrium for the particular examples illustrated in the figure. This figure suggests two main conclusions about equilibrium changes when returns to scale increase. First, when $\gamma < \gamma_m$, the market clearing and zero profit loci, f and g, cross exactly once for a positive value ρ, so a unique, interior equilibrium generally exists. Second, when $\gamma > \gamma_m$, f and g may cross more than once. Thus, $\gamma_m \equiv \alpha/\varepsilon$ is a threshold value of γ, below which there is a unique interior equilibrium, and above which multiple interior equilibria may occur.

Lemmas 1 and 2 in Appendix guarantee that the location of the step in g lies to the left of the zero of f, that is, that $\rho_L < \rho_M$. However,the step in g can lie above the asymptote of f, as in the top two panels of figure 1, or below, as in the bottom two panels. That is, ρ_L can be larger or smaller than ρ_m. The top two panels of figure 1, (7) and (b), describe the case when the asymptote of f lies to the left of the step in g, i.e., $\rho_m < \rho_L$, while the bottom two panels, (c) and (d), describe the opposite case.

Lemma 3 in Appendix E provides necessary and sufficient conditions for $\rho_L > \rho_m$. Informally, lemma 3, establishes that we have $\rho_m < \rho_L$ if commuting costs are high *or* the demand for commercial land is sufficiently large relative to the demand for residential land. Conversely, if commuting costs are low *and* the demand for commercial land is low, then $\rho_m > \rho_L$. We will see below that equilibrium depends importantly on which condition holds.

[7]Barring corner outcomes, L_i is strictly between zero and one. As a result, L_i^γ is decreasing in γ and $\partial L_i^\gamma / \partial L_i$ is increasing in γ.

4 Constant returns to scale

To begin, we consider a benchmark case when production is constant returns to scale, and there are no spillovers or first nature advantages. The following proposition characterizes the unique spatial equilibrium.

Proposition 3 *Suppose $\gamma = \delta = 0$ and $b = c = d = 1$. Then, there exists a unique equilibrium. The equilibrium has the following properties: (i) the equilibrium is interior; (ii) $0 < \omega^*, \rho^* < 1$; (iii) if $\psi > 1$, then $\ell^* > 1$, i.e., the equilibrium employment pattern is bell-shaped; (iv) as $\varepsilon \to \infty$ the equilibrium becomes flat.*

Proof: See Appendix F.

Recalling the definitions of ρ and ω, $0 < \rho^* < 1$ and $0 < \omega^* < 1$ mean that equilibrium land rent is higher in the center and the wage lower. This is surprising. Even without first nature advantages, spillovers, or returns to scale, equilibrium agglomeration still occurs.

To understand why this occurs, consider the problem of a household faced with a choice of location and residence when wages and rents are the same in all locations. If we let $V = W/R^{\beta}$, then using (2), such a household's discrete choice problem is

$$\max_{ij} \left\{ \begin{array}{ccc} z_{-1,-1}V, & \frac{z_{-1,0}}{\tau}V, & \frac{z_{-1,1}}{\tau^2}V \\ \frac{z_{0,-1}}{\tau}V, & z_{0,0}V, & \frac{z_{0,1}}{\tau}V \\ \frac{z_{1,-1}}{\tau^2}V, & \frac{z_{1,0}}{\tau}V, & z_{1,1}V \end{array} \right\}.$$

This is the standard way of stating a discrete choice problem, except that we arrange the nine choices in a matrix so that the row choice corresponds to a choice of residence and choice of column to a choice workplace.

Suppose we restrict households to all choose a central residence. Because the distribution of idiosyncratic tastes is identical for all location pairs, the average payoff for a household at a central residence is

$$E\left(\max\left\{\frac{z_{0,-1}}{\tau}V, z_{0,0}V, \frac{z_{0,1}}{\tau}V\right\}\right) = \Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(1+\frac{2}{\tau^{\varepsilon}}\right)^{1/\varepsilon} V. \tag{27}$$

If, instead, we restrict households to choose a peripheral residence, then

$$E\left(\max\left\{z_{-1,-1}V, \frac{z_{-1,0}}{\tau}V, \frac{z_{-1,1}}{\tau^2}V\right\}\right) = \Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(1+\frac{1}{\tau^{\varepsilon}}+\frac{1}{\tau^{2\varepsilon}}\right)^{1/\varepsilon} V. \tag{28}$$

Because $\tau > 1$, it follows that the average payoff for a household in a peripheral residence is less than an average household in a central residence. By symmetry, exactly the same logic applies to the choice of employment. This occurs despite the fact that wages and rents are the same in all locations. In this sense, this discrete choice problem creates an *average preference* for residence and employment in the central location. Proposition 3 tells us that in equilibrium, these preferences are capitalized into lower central wages and higher central rents. While our model is simple, this phenomena appears to be general. If we exclude empirically uninteresting geographies like circles, most remaining location sets have a center in the sense of this example.

It is tempting to think that the average preference for central work and residence is a response to commuting costs. This is not correct. From Proposition 3, when $\varepsilon \to \infty$, and preferences become homogenous, we obtain the perfectly flat equilibrium in which each location is in autarky.[8] This is the outcome we expect in an economy where neither first nature, nor spillovers, nor IRS operates. Inspection of (27) and (28) shows why this occurs. When $\varepsilon \to \infty$, (27) and (28) are identical. Commuting costs alone are not sufficient to create an average preference for central work or residence.

Against the two centralizing forces of average preferences are set two centrifugal forces. There is twice as much land in the periphery as the center. Because land contributes to utility and productivity, the scarcity of central land incentivizes the movement of employment and residence to the periphery. Whether the center ends up relatively specialized in residence or employment depends on which of the two activities has the highest demand for land, and this activity will locate disproportionately in the land abundant periphery.

Proposition 3 makes this intuition precise. Agglomeration of production occurs at the center when $\psi > 1$ holds. Using the definition of ψ (19) we have

$$\frac{\alpha\beta}{1-\alpha} < \frac{1+\varepsilon}{\varepsilon}.$$

Recalling that α is the labor share in production and β the land share of consumption, we see that $\alpha\beta$ is the share of every dollar of firm revenue used

[8]When we assume that households are homogeneous, a flat equilibrium in which nobody commutes always exists. However, other equilibria may also exist according to the values of some parameters. The proof is available upon request from the authors.

for residential land. Thus the left hand side of this expression is the ratio of the indirect demand for residential land in production to the demand for land in production. As this ratio increases, the importance residential of land increases relative to commercial land, and we expect to see residential activity concentrated in the periphery where land is abundant and production in the center where it is scarce. This is precisely what Proposition 3 says.

5 First nature

We now turn attention to conventional sources of agglomeration. We begin with an examination of first nature productivity advantages. To isolate the role of first nature productivity advantages, we consider the case when production is constant returns to scale, $\gamma = 0$, and there are no spillovers, $\delta = 0$, although we impose no restrictions on residential and workplace amenities.

The following proposition characterizes the corresponding unique spatial equilibrium.

Proposition 4 *Suppose $\gamma = \delta = 0$. Then, there exists a unique equilibrium; this equilibrium is interior. The equilibrium labor pattern becomes more concentrated as c increases. There exists a threshold level $\bar{c} > 0$ such that the equilibrium labor pattern is: (i) $(1/2, 0, 1/2)$ when $c \searrow 0$; (ii) U-shaped, (L_1, L_0, L_1) with $0 < L_0 < L_1$, when $0 < c < \bar{c}$; (iii) flat $(1/3, 1/3, 1/3)$ when $c = \bar{c}$; (iv) bell-shaped, (L_1, L_0, L_1) with $L_0 > L_1 > 0$, when $\bar{c} < c < \infty$; (v) $(0,1,0)$ when $c \nearrow \infty$.*

Proof: See Appendix G.

Although we make no assumption about the values of b and d, the threshold value $\bar{c}$ depends on these parameters.

Proposition 4 shows that any symmetric employment pattern may be sustained as a spatial equilibrium for some value of c, and that employment concentrates where first nature productivity is greatest. This seems unsurprising, but requires two comments. First, Proposition 4 describes the relationship between first nature productivity and equilibrium employment patterns. It is silent about the residential pattern and commuting behavior. We conjecture that choosing b and d in addition to c would also identify these other endogenous quantities. Second, Proposition 4 resembles Proposition 2 in Ahlfeldt *et al.* (2015). However, our result slightly extends the Ahlfeldt *et al.* result by mapping out the relationship between first nature advantages and equilibrium outcomes.

6 Spillovers and increasing returns

We now consider a city where, in addition to arbitrary first nature advantages, increasing returns or to scale or productivity spillovers operate at low levels. More precisely, we compare the impact of local increasing returns and technological spillovers in the vicinity of $\gamma = 0$ and $\delta = 0$ for arbitrary first nature productivity advantages. In doing so, we restrict attention to empirically relevant values of the two parameters.[9]

Proposition 5 *Consider an economy with arbitrary first nature productivity. Then,*

(i) There exists a unique threshold $\hat{c} > 0$ such that increasing γ slightly above 0 increases central employment if $c > \hat{c}$. When $c < \hat{c}$, increasing γ slightly above 0 increases peripheral employment.

(ii) There exists a unique threshold $\widetilde{c} > 0$ such that increasing δ slightly above 0 increases central employment if $c < \widetilde{c}$. When $c > \widetilde{c}$, increasing δ slightly above 0 increases peripheral employment.

Proof: See Appendix H.

The first part of the proposition is expected. Increasing returns magnify the initial first-nature productive advantage of a location. Increasing γ raises the relative TFP $a(\gamma)$ if and only if the equilibrium under constant returns is such that $\ell > 1$. Thus, if first nature concentrates employment in the center, IRS increases this concentration. Furthermore, when $c > \hat{c}$, the greater the relative initial advantage is, the greater is the impact of increasing returns.

The second part of Proposition 5 is more surprising. If first nature advantage leads to sufficiently high central employment, then spillovers encourage peripheral employment because the productivity gains generated by the center are large. Conversely, if first nature advantages sufficiently high peripheral employment, then spillovers increase central employment. The intuition behind this result is clear. If first nature advantage leads to the concentration of employment in either the center or the periphery, then spillovers allow firms to benefit from this concentration of employment without competing for space in the most productive location.

[9] Estimates of agglomeration economies suggest that IRS and spillover effects, γ and δ, are much less than one (Rosenthal and Strange, 2020).

Summarizing, we have (1) if $c < \min\{\widetilde{c},\overline{c}\}$ or $c > \max\{\widetilde{c},\hat{c}\}$, changes to increasing returns and spillovers have opposite effects on the employment distribution, and (2) if $c \in (\min\{\widetilde{c},\hat{c}\}, \max\{\widetilde{c},\hat{c}\})$, changes to increasing returns and spillovers have similar effects on the employment distribution. That is, we expect qualitatively similar comparative statics for IRS and spillovers if and only if first nature advantages do not strongly favor the center or the periphery.

Proposition 5 establishes that conflating 'IRS' and 'spillovers' as 'agglomeration forces' is wrong on two counts. First, IRS and spillovers are distinct economic forces which may have different effects on an equilibrium city. Second, IRS and spillovers need not act as agglomeration forces at all.

Note that Proposition 5 does not contradict the conclusion in, e.g., Fujita and Ogawa (1982), that productivity spillovers can lead to an equilibrium where employment is concentrated in a central disk center and commuting workers reside in the peripheral ring. Such results arise in the absence of first nature advantages when spillovers are sufficiently large. In contrast, Proposition 5 requires the presence of first nature advantages and that spillovers be sufficiently small.

Because of the difficulty of analyzing transcendental equilibrium conditions, and because of the availability of results in the literature relevant to this part of the parameter space (e.g., Fujita and Ogawa (1982), Lucas and Rossi-Hansberg (2002) and Berliant *et al.* (2002)) we now turn our attention to the analysis of equilibrium in economies where returns to scale operates.

7 Increasing returns

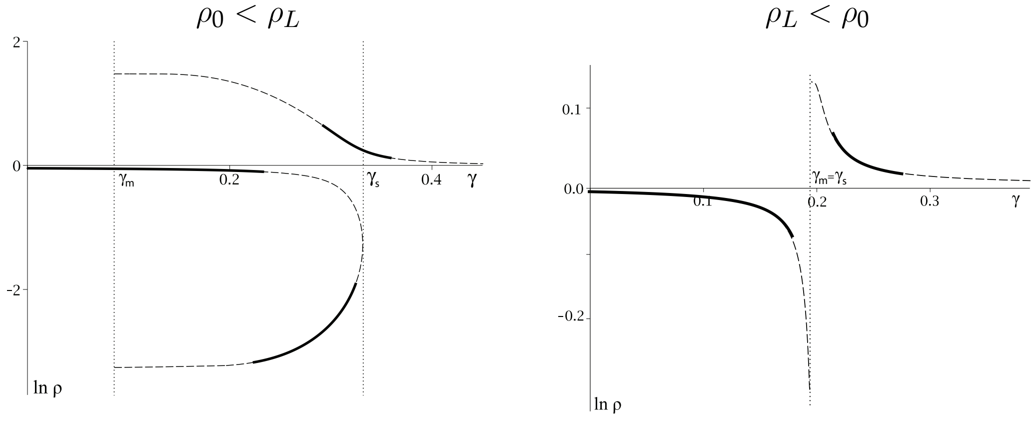
We now turn attention to the role of increasing returns to scale. To focus attention on returns to scale, we rule out spillovers, and ask what happens as γ increases.

The following proposition characterizes the set of equilibria under increasing returns, $\gamma > 0$, when spillovers do not operate.

Proposition 6 *Assume $\delta = 0$. If $\gamma > 0$, then (i) an interior equilibrium always exists; (ii) there exist two corner equilibria. These corner equilibria are such that the employment patterns are given, respectively, by* $(0,1,0)$ *and* $(1/2,0,1/2)$. (iii) *In both corner and interior equilibria, each location hosts a positive mass of residents.*

Proof: See Appendix I.

Figure 2: Equilibrium correspondence between ρ and γ.



Notes: *In both panels the x-axis is γ and the y-axis is $\ln \rho$. The left panel illustrates all interior equilibria as γ varies when $\rho_0 < \rho_L$. The right panel shows the case where $\rho_L < \rho_0$. Solid lines indicate stable equilibria and dashed lines indicate unstable equilibria, where stability is defined as in Section 8.*

Part (i) of Proposition 6 is exactly what we would expect from inspection of figure 1; throughout the range of increasing returns, the market clearing locus, f, and the zero profit locus, g, always have at least one interior intersection. Likewise, the Fréchet distribution for the support of the z_{ij} in the indirect utility function (2) is unbounded, so we expect that every location will always have residents, as required by part (iii).

Part (ii) of Proposition 6 is surprising. In spite of the unbounded support of the z_{ij}, not every residence-workplace pair must be populated in equilibrium. An inspection of the production technology in equation (10) solves the puzzle. If the amount of land devoted to production is zero in a location, then the marginal product of labor is also zero. With a competitive labor market this requires that the wage also be zero, and hence, by inspection of (2), that the utility of any worker at this location be zero. These corner equilibria appear to depend sensitively on the multiplicative structure of our Cobb-Douglas production function.

With existence established, we now turn to a characterization of equilibrium as γ increases from zero. We begin by introducing terminology to describe the three important domains of returns to scale. We establish below that these ranges are associated with qualitatively different equilibrium behavior.

Definition 2 *Increasing returns to scale are: (i) weak if $0 < \gamma < \gamma_m \equiv \alpha/\varepsilon$; (ii) moderate if $\gamma_m \leq \gamma \leq \gamma_s$; or (iii) strong if $\gamma > \gamma_s$.*

As we describe above, $\gamma_m = \alpha/\varepsilon$ is the value of γ at which the zero profit locus switches from being an increasing to a decreasing function of ρ. Because $\gamma_m \to 0$ as $\varepsilon \to \infty$, it follows that the population must be heterogeneous for weak returns to scale to occur.

Like γ_m, γ_s is also a threshold value. Anticipating our results below, figure 2 illustrates the equilibrium correspondence between $\ln \rho$ and γ. To define γ_s, we solve the equilibrium condition $f(\rho) = g(\rho; \gamma)$ for γ to get

$$\gamma(\rho) = \frac{\log\left(\rho^{-b} f(\rho)\right)}{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}\right)}.$$

The function $\gamma(\rho)$ may be obtained by reversing the axes in figure 2(a). We define γ_s as the global maximizer of $\gamma(\rho)$ subject to $\rho_m \leq \rho \leq 1$. Considering again the correspondence in figure 2(a), we see that γ_s is also the threshold value of γ at which the two lower equilibrium branches merge and disappear.[10]

Weak increasing returns

The next proposition describes the equilibrium when increasing returns to scale are weak.

Note that, without a restriction on c, Proposition 4 implies that any configuration of employment may emerge as an equilibrium outcome. Therefore, to isolate the effect of γ, in what follows, we assume that there is no first nature production advantage ($c = 1$), although we usually do not restrict relative amenities b and d.

Proposition 7 *Assume $\delta = 0$ and $c = 1$. (i) If $0 < \gamma < \gamma_m$, then there is a unique interior equilibrium. (ii) Furthermore, if $\psi > 1$ holds, then the equilibrium employment pattern is bell-shaped and*

$$\frac{d\ell^*}{d\gamma} > 0, \qquad \frac{d\rho^*}{d\gamma} < 0 < \frac{d\omega^*}{d\gamma}.$$

Proof: See Appendix J.

In the region of weak IRS, equilibrium is largely determined by the same forces that operate when the technology is constant returns to scale. That is, the average preferences for central work and residence draw activity into the center

[10]Interestingly, we can show that $\gamma_m = \gamma_s$ when $\rho_m \geq \rho_L$ and $\gamma_m < \gamma_s$ when $\rho_m < \rho_L$. Therefore, the region of moderate increasing returns to scale does not exist unless the step in the zero profit locus is to the right of the asymptote of the market clearing locus, i.e., $\rho_m < \rho_L$.

and the relative abundance of peripheral land pulls it out. As scale economies increase, the central location becomes increasingly attractive for employment, the central land price capitalizes higher central productivity, and the central wage rises in response to the increase in the marginal product of labor.

The comparative statics in Proposition 7 holds whenever $0 < \gamma < \gamma_m$. The generality of this result conceals the fact that distinct equilibrium regimes arise when $\rho_m < \rho_L$ and $\rho_L < \rho_m$.

When $\rho_m < \rho_L$, high commuting costs encourage households to work where they live or land hungry production faces pressure to disperse to the periphery (or both). An equilibrium in such an economy has low levels of commuting and dispersed production. Thus, when $\rho_m < \rho_L$, low levels of IRS do not lead to equilibrium cities where employment or residence is highly concentrated in either the center or periphery. Panel (a) of figure 1 illustrates this case.[11]

In contrast, when $\rho_L \leq \rho_m$, low commuting costs allow households to separate work and residence locations in response to a small wage premium, and productivity is not sensitive to the relatively abundant land of the periphery. In this case, IRS compounds the average preference for central employment to concentrate employment in the center, and households are able to cheaply disperse their residences to the land abundant periphery. An equilibrium in such an economy involves concentrated employment and high levels of commuting. Thus, when $\rho_L < \rho_m$, low levels of IRS lead to equilibrium cities where employment is highly concentrated in the center and residence in the periphery. The monocentric city arises endogenously.[12]

Figure 2 shows the equilibrium relationship between γ and $\ln \rho$ for numerical examples satisfying $\rho_m < \rho_L$ in panel (a) and $\rho_L < \rho_m$ in panel (b). In both panels, the x-axis is γ and the y-axis is $\ln \rho$. Both figures show all interior equilibria, but not the corner equilibria required by Proposition 6. Both figures anticipate our analysis of stability in Section 8 and indicate stable equilibria with a solid line and unstable equilibria with a dashed line. Consistent with our results in

[11] More formally, as γ approaches γ_m, the zero profit locus converges to an increasing step function with the step at ρ_L. Hence, the curves f and g must cross near ρ_L as γ increases towards γ_m. Because $\rho_m < \rho_L$, it follows that this intersection must occur when f is away from its asymptote at ρ_m. Therefore, the equilibrium value of ω grows with γ but remains bounded.

[12] More formally, when $\rho_L \leq \rho_m$, as γ approaches γ_m, the intersection of f and g occurs near the asymptote of f. As a result, the value of ω at which the two curves intersect becomes large.

Propositions 3 and 7, figure 2 shows that ρ decreases from near one as γ increases in a neighborhood of zero. As γ increases towards γ_m, we see that ρ continues to decrease. Comparing panels (a) and (b) we see the two distinct equilibrium regimes that arise as γ increases toward the threshold of the weak IRS domain when $\rho_m < \rho_L$ and $\rho_L < \rho_m$.

Moderate increasing returns

When $\gamma > \gamma_m$, the market clearing locus, f, remains unchanged, but the zero profit locus, g, changes from an increasing to a decreasing function. When f and g are both decreasing, they need not cross at all, and may cross more than once. Thus, zero or many equilibria are possible. Proposition 6 establishes that an equilibrium exists. Figure 1 suggests that for intermediate values of γ, that f and g cross one or three times. The following proposition formalizes this intuition.

Proposition 8 *Assume $\delta = 0$ and $c = 1$. If γ is slightly above γ_m, then*

i. if $\rho_m < \rho_L$, there exist two interior equilibria, (ρ_1^*,ω_1^*) and (ρ_3^*,ω_3^*), such that, $\omega_1^* > 1 > \omega_3^*$ and $\rho_1^* < 1 < \rho_3^*$, as well as a third interior equilibrium. As $\gamma \searrow \gamma_m$, the first two equilibrium employment patterns converge to $(1/2,0,1/2)$ and $(0,1,0)$, while the third equilibrium is interior.

ii. if $\rho_m \geq \rho_L$, there exists a unique interior equilibrium (ρ^*,ω^*) such that, $\omega^* < 1$ and $\rho^* > 1$. Furthermore, as $\gamma \searrow \gamma_m$, the equilibrium employment pattern converges to $(1/2,0,1/2)$.

Proof: See Appendix K.

The logic underlying part (i) of Proposition 8 is visible in panel (b) of figure 1. The medium blue line describes the case of moderate increasing returns. In this case, g crosses f three times. At the first intersection point, we have $\rho_1^* < 1$ and $\omega_1^* > 1$; at the second, we see that ρ_2^* approaches ρ_L as γ decreases toward γ_m; at the third intersection point, we have $\rho_3^* > 1$ and $\omega_3^* < 1$. The value ω_1^* (resp., ω_3^*) in turn requires that employment occurs primarily in the center (resp., periphery).

The light blue line in panel (b) describes g when γ is just above γ_m. As gamma approaches this threshold, for one of the two new equilibria ω grows without bound (and occurs outside the frame of the figure) while ω approaches zero in the other equilibrium. That is, just above the threshold, these two equilibria approach corner patterns where all employment is either central or peripheral.

The corresponding logic for part (ii) of Proposition 8 is visible in panel (d) of figure 1. As in panel (b), the medium blue line describes the case of moderate increasing returns. In this case, g crosses f only once. The light blue line in panel (b) describes g when γ is just above γ_m. As gamma approaches this threshold, the single intersection of f and g occurs at progressively larger values of ω as g converges to a decreasing step function. In the limiting case, as γ approaches γ_m from above, the single equilibrium occurs when all employment is concentrated in the periphery. Figure 2 summarizes the results of Proposition 8.

That part (i) of Proposition 8 establishes the emergence of multiple equilibria seems unsurprising. We expect sufficiently strong IRS to give rise to multiple equilibria. Part (i) also describes an equilibrium branch which continues behavior form the weal IRS case. This is the central, interior equilibrium branch in panel (a) of figure 2. That the logic governing behavior in the case of weak IRS should sometimes survive a small increase in γ above γ_m also seems unsurprising.

Two results in Proposition 8 are less expected. First, we see in part (ii) that multiple equilibria need not emerge as returns to scale increase. When commuting costs are low and land is less productive, i.e. $\rho_m > \rho_L$, there is only a single equilibrium when γ is weak or moderate. Thus, increasing returns is necessary for multiple equilibria, but not sufficient.

Second, Propositions 7 and Proposition 8 together imply a discontinuous change in the only possible equilibrium city when $\rho_L < \rho_m$ and γ varies around γ_m. This discontinuity is clearly visible in panel (b) of figure 2. It is equally clear that this discontinuity follows from the fact that the zero profit curve switches from an increasing to a decreasing step function around this singular point.

These results require two comments. First, notice that Proposition 8 characterizes equilibrium just above the threshold separating weak and moderate increasing returns, γ_m. It is natural to expect that the behavior we observe near γ_m persists throughout the full range of moderate increasing returns, as in the example illustrated in Figure 2. In fact, we cannot rule out the possibility of more complicated equilibrium behavior for values of γ just below γ_s, although we did not find a counter example to contradict the conjecture that the results of Proposition 8 hold throughout the range of moderate increasing returns.

Second, Proposition 8 establishes qualitatively different equilibrium behavior around $\gamma = \gamma_m$ when $\rho_m < \rho_L$ and conversely. Recalling that $\rho_L < \rho_m$ describes the case when commuting costs and the land share of production are both low, we expect that the location of employment to be more sensitive to changes in returns to scale than in the opposite case. This intuition is consistent with our finding in Proposition 8.

Strong increasing returns

Careful inspection of figure 1 shows that the market clearing and zero profit curves cross only once and that a single interior equilibrium persists when γ is larger than a threshold γ_s, regardless of whether $\rho_m < \rho_L$ or $\rho_L < \rho_m$. The following theorem extends and makes precise this intuition.

Proposition 9 *Assume $\delta = 0$ and $c = 1$. If $\gamma > \frac{1+\varepsilon}{(1-\beta)\varepsilon} - \alpha \geq \gamma_s$, then there exists a unique interior equilibrium. Furthermore, the equilibrium employment pattern gets flatter as γ rises and converges to the uniform pattern when $\gamma \to \infty$.*

Proof: See Appendix L.

As γ increases beyond γ_m, only a single interior equilibrium persists. This equilibrium involves a moderate value of ω, so that employment is more or less evenly distributed across the three locations. In this region of the parameter space, as returns to scale increase, it leads households to distribute themselves more uniformly, and in the limiting case, to a perfectly flat equilibrium.

Inspection of figure 1 makes it clear why this result occurs. As γ increases beyond γ_m, the zero profit curve, g, diverges from the step function $g(\rho,\gamma_m)$ and as this occurs, the intersection of f and g occurs at values of ρ that are progressively nearer to one.

Figure 2 summarizes this result. As γ increases beyond the threshold γ_s, regardless of the relative magnitude of ρ_m and ρ_M, we see that $\ln \rho$ moves back towards zero, so that increases in γ beyond the threshold of the strong increasing returns equalizes central and peripheral rents.

This result is novel and surprising in two regards. First, as expected Proposition 7 shows that for low levels of returns to scale, increases in γ lead to increased concentration of employment in the center. However, Proposition 9 shows that beyond a certain point, this relationship reverses and further increases in returns to scale lead to a more equal distribution of employment. Thus, IRS is not an agglomeration force over its entire possible range. At some point, it causes employment to disperse. This seems surprising. Second, even in environments where IRS leads to multiple equilibria (as expected), sufficiently high returns to scale leads, once again, to unique equilibria. The ability of increases in returns to scale to eliminate multiple equilibria is also novel and surprising.

To our knowledge, Proposition 9 is new to the literature. While it has long been understood that increasing returns to scale could lead to multiple equilibria, the idea that sufficiently high increasing returns leads, once again, to a unique equilibrium is novel. Even more surprising, this equilibrium involves more dispersion of production as the degree of increasing returns rises.

Note that dispersion of employment in response to increases in returns to scale also arises when returns to scale are moderate. We see in Proposition 8(i) that along the equilibrium branches where employment is highly concentrated in the center or the periphery employment concentration decreases in response to increases in γ.

To understand this result consider the zero profit condition given in equation (14). As returns to scale increase, so does TFP. If profits are to remain constant, some other quantity must adjust. The candidates are wages, rents and employment (on which TFP also depends). The need for adjustment is compounded if labor migrates to more productive places and further increases their productivity. It is clear that if land rent and wages go up, this offsets the increase in productivity and restores the zero profit condition. However, Proposition 9 shows that equilibrium increases in wages and rents are not always sufficient to preserve the zero profit condition. Once returns to scale are sufficiently high, restoring the zero profit condition is accomplished by dispersing employment to reduce productivity.

8 Stability

We have seen that multiple equilibria can arise in much of the parameter space. In such cases, it is common to appeal to stability as a selection device. This leads to the question of how to define stability.

One candidate, particularly relevant for the literature on quantitative spatial models, is to say that an equilibrium is stable if an iterative process will converge to it. Formally, if equilibria are defined by $f(\rho) = g(\rho)$ then equilibria are fixed points of $h(\rho) = \rho$, for $h(\rho) \equiv f^{-1}(g(\rho))$. In this case, it is well known that an iterative process will find a fixed point ρ^* if and only if $|h'(\rho^*)| < 1$. Surprisingly, this notion of stability is not well defined. There are two problems.

Recalling that our fixed point is defined by $f(\rho) = g(\rho)$, it is clear that there are two alternative ways of stating the fixed point problem. First, as $\rho = h(\rho) \equiv f^{-1}(g(\rho))$, and alternatively, as $\rho = \widetilde{h}(\rho) \equiv g^{-1}(h(\rho))$. Both representations have the same solutions, but their stability properties are opposite. It is straightforward to show that for any solution of this problem, $|h'(\rho^*)| < 1$ if and only if $|\widetilde{h}'(\rho^*)| > 1$. Thus, the iterative stability of any given solution to the fixed point problem that defines equilibrium depends sensitively on arbitrary decisions about the representation of the fixed point problem. Thus, iterative stability is not well defined and iterative methods cannot be expected to find all of equilibria of an economy when multiple equilibria exist.

To understand the second problem, observe that for any $0 < \theta < 1$, the equation $h(\rho) = \theta\rho + (1-\theta)\rho$ also defines solutions of $f(\rho) = g(\rho)$, so that fixed points of $\widetilde{h}(\rho) = [(h(\rho) - (1-\theta)\rho)/\theta] = \rho$ are also solutions of $f(\rho) = g(\rho)$. However, the stability properties of this second equation may be different from the original. By choosing θ sufficiently small, we guarantee that $|\widetilde{h}'(\rho^*)| > 1$. Thus, this second argument leads to the same conclusion as the first. Iterative stability is not well defined and iterative methods cannot be expected to find all of equilibria of and economy when multiple equilibria exist. Note that the standard practice of starting the iterative algorithm from a variety of initial conditions does not respond these problems.

These problems with computational methods motivate our interest in analytical methods and solutions. The discussion above demonstrates that we cannot rely on numerical methods to find all of the equilibrium configurations of an economy if we do not at least know how many equilibria there are. This leads us, in turn, to depart from the practice of the modern quantitative literature of studying models with arbitrary geographies, in favor of our more tractable three location linear city.

An alternative approach to stability involves specifying state variables for the economy and adjustment process for these state variables. This approach to stability is well know (e.g., Krugman 1991). In the context of our problem, symmetry implies that we must determine the values of three variables to obtain the equilibrium outcome. For example, it is sufficient to know L_0, M_0, and s_{00} to determine all the s_{ij}, and hence the vector $\{\mathbf{M},\mathbf{L},\mathbf{H},\mathbf{N},\mathbf{W},\mathbf{R}\}$. To implement notion of stability, we must specify an adjustment process describing how L_0, M_0 and s_{00} respond to a perturbation. Stability is then well defined in the resulting dynamic system. This approach is subject to two problems. First, it is likely to be intractable. Second, it must rest on ad hoc descriptions of the adjustment process, and we suspect that the stability of any particular equilibrium is likely to be sensitive to these assumptions.

These difficulties lead us to a more game-theoretic notion of stability. In the spirit of trembling hand perfection, we say that an equilibrium is stable if households want to return to the equilibrium when an arbitrarily small measure of them are displaced. This definition of stability has three advantages. First, like our model, it is static and does not require an explicit description of time. Second, and unlike the other candidate definitions of stability, it has explicit behavioral foundations. Third, as we demonstrate, it is tractable.

Let ij and kl be two arbitrary location pairs; $ij = kl$ (location pairs are equal) when $i = k$ and $j = l$ hold simultaneously and distinct otherwise. We say that an equilibrium is *unstable* if, for some $ij \neq jk$, for any arbitrarily small $\Delta > 0$, there is a subset of individuals of mass Δ who strictly prefer the location pair kl, which differs from their utility-maximizing pair ij, when a perturbation moves them all to kl. In other words, the subset of individuals who have been moved away from ij do not want to move back. Otherwise, the equilibrium is *stable*.

The key issue is to determine the subset of individuals to use to check whether the equilibrium is unstable. In what follows, we assume that this subset is formed by individuals whose types are close to those of an individual indifferent between her equilibrium pair ij and another location pair kl.

Consider an equilibrium commuting pattern $\mathbf{s}^* \equiv (s^*_{ij})$, which could be interior or corner, and two location pairs, ij and kl, such that $ij \neq kl$ and $s^*_{ij} > 0$. We say that an individual $\nu \in [0,1]$ is *indifferent between ij and kl* if and only if

$$V^*_{ij}(\nu) = V^*_{kl}(\nu) \geq V^*_{od}(\nu), \tag{29}$$

for every location pair od such that $od \neq ij$ and $od \neq kl$. Lemma 5 in Appendix M establishes that such an individual always exists.

With this definition in place, we can now state our definition of stability formally.

Definition 3 *Consider an arbitrarily small subset of individuals of measure $\Delta > 0$ who choose ij and have types close to $\mathbf{z}(\nu) \in S_{ij}$ where ν is indifferent between ij and $kl \neq ij$. If this individual is strictly better off when she and her neighboring individuals are relocated from ij to kl, the spatial equilibrium is unstable. Otherwise, the spatial equilibrium is stable.*

The motivation for this definition is as follows. If the relocation of a small group of almost indifferent individuals from ij to kl makes the indifferent agent strictly better off, then, by continuity there is a non-negligible subset of individuals who strictly prefer kl to ij. Hence, these individuals will never switch back to ij. On the contrary, if the indifferent individual never becomes strictly better off for any small subset, no other individual strictly prefers a different location pair. Hence, all the individuals will be willing to switch back to ij.

By relocating a small subset of individuals from ij to kl, the commuting pattern $\mathbf{s}$ becomes different from the equilibrium pattern $\mathbf{s}^*$. Hence, for our definition of stability to make sense, we must be able to compare the equilibrium and off-equilibrium utility levels. For this to be possible, we must determine the conditional equilibrium vectors of wages and land rents $\overline{\mathbf{W}}(\mathbf{s})$ and $\overline{\mathbf{R}}(\mathbf{s})$. We show in Appendix N that, for $\alpha > 1/2$, these vectors exist, are unique and continuous in $\mathbf{s}$.

This definition of stability equips us to study the stability of the equilibria identified in Proposition 6. We start with corner equilibria and show in Appendix O that these equilibria are unstable. This result is easy to understand. Consider the agglomerated corner equilibrium $\mathbf{L}^* = (0,1,0)$. No single individual wants to move to, say, location 1 because her marginal productivity would be zero. This is why (0,1,0) is an equilibrium employment pattern. By contrast, when a small subset of workers happens to be at $j = 1$, the marginal product of labor, and therefore the incomes, of individuals whose tastes are close to those of the indifferent individual are high. As a consequence, they do not want to move back to location 0. If we rely on stability to select among multiple equilibria, it means that we can ignore the corner equilibria. This result has the potential to greatly simplify quantitative exercises based on this family of models.

By Proposition 6, interior equilibria always exist. In appendix P, Proposition 10 provides a simple test for checking the stability of any interior equilibria. Applying this test to the examples illustrated in figure 2 allows us to determine the stability of each possible equilibrium for the relevant parameter values. In figure 2, we indicate stable equilibria with a heavy solid line, and unstable equilibria with a lighter dashed line.

While our results do not permit general conclusions about the stability of equilibria, they demonstrate that stable equilibria need not exist and that multiple stable equilibria are also possible.

9 Conclusion

Understanding how people arrange themselves when they are free to choose work and residence locations, when commuting is costly, and when some economic mechanism rewards the agglomeration of employment is one of the defining problems of urban economics. We address this problem by combining a discrete choice model of location, the stylized geographies of classical urban economics, and a production function that allows for first nature advantages, IRS, and productivity spillovers. We provide a complete description of equilibria in much the parameter space.

Besides accounting identities, an equilibrium must satisfy two main conditions: all households choose their most preferred workplace-residence pair and profits must be zero everywhere. Of these two, the first is familiar from widely used discrete choice models of spatial equilibrium. The second is less well studied and has two surprising implications. First, equilibrium agglomeration of employment is first increasing and then decreasing in the strength of returns to scale. When increasing returns to scale are strong enough, the zero profit condition is preserved, in part, by dispersing employment. Second, productivity spillovers can act to disperse employment. Productivity spillovers allow firms to benefit from high productivity locations without paying the rent and wage premium required to locate in them. It is enough to be near. Of the three conventional foundations for economic agglomerations that we consider, only the comparative statics for first nature that behave as expected: employment concentrates where first nature productivity is greater.

Despite its wide use, our conventional description of preference heterogeneity implies a previously unnoticed foundation for agglomeration. A population of households with heterogenous preferences over workplace-residence pairs has an average preference for central work and residence. Absent any property of production that rewards the concentration of employment, a city comprised of such households has denser central employment and residence.

Even in a stylized geography, the relationship between economic fundamentals and equilibrium is complicated. This is true throughout the parameter space, but particularly in the region of moderate returns to scale. As we see in figure 2, it is in this region where multiple interior equilibria arise, where equilibrium comparative statics can be discontinuous, and where increasing to returns to scale begins to disperse employment. Given this, it is natural to ask whether such values of returns to scale are empirically relevant.

The relevant threshold level of IRS is $\gamma_m = \alpha/\varepsilon$, the lower boundary of the region of moderate returns to scale. In a modern economy, the labor share of production, α, is about 0.6, while the range of commonly used estimates for ε is about [5,7]. Taking the ratio of these values, we have γ_m in [0.085,0.12]. Estimates of the wage elasticity of population for modern, developed country cities that control for sorting and first-nature productivity are often around 0.05. However, the raw correlation between wages and density is larger, as are estimates for developing countries. This suggests that γ_m will sometimes lie in an empirically relevant part of the parameter space.

The implications for these results suggest a number questions for further research. For much of the past generation, a principal activity of urban economists has been the estimation of 'agglomeration economies', the relationship between a measure of productivity and the size or density of city employment. Our results suggest that this empirical relationship describes a reduced form for a complex interaction between first nature, IRS, productivity spillovers, and preference heterogeneity. Refining our understanding of the foundations of agglomeration requires research designs that will permit us to distinguish these fundamentals. The possibility of multiple equilibrium in an empirically relevant region of the parameter space raises further questions about the interpretation of observed reduced form relationships between productivity and density.

Our results also have implications for research based on quantitative spatial models. As a rule, such models often share many features with the one considered here, and so may be expected to exhibit at least some of the same complicated behavior. The possibility of complex behavior in a neighborhood of the boundary between the weak and moderate returns to scale together with the empirical relevance of this threshold suggests that efforts to investigate the possibility of multiple equilibria are appropriate. As we have shown, fixed point algorithms are not well suited to this sort of search for equilibria, and so an investigation of multiple equilibria appears to require new techniques. Taking as given that an analytic characterization is infeasible, one possibility is to reformulate the equilibrium conditions in different ways, and to apply a fixed point algorithm to each such formulation.

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Appendix

A. Proof of Proposition 1

As commuting flows (7) are uniquely determined by ρ and ω, it suffices to express the equilibrium patterns — the labor pattern (L_0,L_1), the residential population pattern (M_0,M_1), the housing pattern (H_0,H_1), the commercial land-use pattern (N_0,N_1), the wage pattern (W_0,W_1), and the land rent pattern (R_0,R_1) — as functions of ρ, ω, and (s_{ij}), or equivalently, as functions of r, w and (s_{ij}).

The land-use pattern. Consider the complementary slackness conditions for producer's profit maximization:

$$\left(\alpha A_i L_i^{\alpha-1} N_i^{1-\alpha} - W_i\right) L_i = 0, \quad \text{with } \alpha A_i L_i^{\alpha-1} N_i^{1-\alpha} - W_i \leq 0 \text{ and } L_i \geq 0,$$

$$\left[(1-\alpha) A_i L_i^{\alpha} N_i^{-\alpha} - R_i\right] N_i = 0, \quad \text{with } (1-\alpha) A_i L_i^{\alpha} N_i^{-\alpha} - R_i \leq 0 \text{ and } N_i \geq 0.$$

These complementary slackness conditions imply that

$$A_i L_i^{\alpha} N_i^{1-\alpha} = \frac{W_i L_i}{\alpha} = \frac{R_i N_i}{1-\alpha}.$$

Hence, the demands for commercial land are given by

$$N_0 = \frac{1-\alpha}{\alpha}\frac{W_0 L_0}{R_0}, \qquad N_1 = \frac{1-\alpha}{\alpha}\frac{W_1 L_1}{R_1}.$$

Using (8), we obtain expressions for the commercial land demands:

$$N_0 = \frac{1-\alpha}{\alpha}\frac{W_0}{R_0}(s_{00} + 2s_{10}), \qquad N_1 = \frac{1-\alpha}{\alpha}\frac{W_1}{R_1}\left[(1+\phi^2)s_{11} + s_{01}\right]. \tag{A.1}$$

Next, plugging the commuting flows (7) into the market demand functions for residential land, $H_i \equiv \sum_j s_{ij}\frac{\beta W_j}{R_i}$, and using $w = W_0/W_1$, we obtain expressions for the residential land demands:

$$H_0 = \beta\frac{W_0}{R_0}\left(s_{00} + 2w^{-1}s_{01}\right), \qquad H_1 = \beta\frac{W_1}{R_1}\left[(1+\phi^2)s_{11} + ws_{10}\right]. \tag{A.2}$$

Computing the ratios, H_i/N_i, for $i = 0{,}1$, and using (9),

$$\frac{H_0}{N_0} = \frac{1-N_0}{N_0} = \eta\frac{s_{00} + 2d\omega^{-\frac{1}{\varepsilon}}s_{01}}{s_{00} + 2s_{10}}, \quad \frac{H_1}{N_1} = \frac{1-N_1}{N_1} = \eta\frac{(1+\phi^2)s_{11} + d^{-1}\omega^{\frac{1}{\varepsilon}}s_{10}}{(1+\phi^2)s_{11} + s_{01}},$$

where η is given by (18). Solving for N_0 and N_1, and using (7), we arrive at expressions (21) – (22) for the land-use patterns.

Wages and land rents. The ratios W_i/R_i, $i = 0{,}1$, are pinned down by combining (A.1) and (A.2) with the land-market clearing conditions (9):

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha}\left[(1+\eta)s_{00} + 2s_{10} + 2\eta w^{-1}s_{01}\right]^{-1};$$

$$\frac{W_1}{R_1} = \frac{\alpha}{1-\alpha}\left[(1+\phi^2)(1+\eta)s_{11} + s_{01} + \eta w s_{10}\right]^{-1}.$$

Restating the zero-profit conditions (14) as

$$R_i = \alpha^{\alpha}(1-\alpha)^{1-\alpha}A_i\left(\frac{W_i}{R_i}\right)^{-\alpha},$$

and plugging the ratios W_i/R_i into the RHSs, we obtain land rents as functions of r, w, and s_{ij}:

$$R_i = \begin{cases}(1-\alpha)A_0\left[(1+\eta)s_{00} + 2s_{10} + 2\eta w^{-1}s_{01}\right]^{\alpha}, & i = 0;\\ (1-\alpha)A_1\left[(1+\phi^2)(1+\eta)s_{11} + s_{01} + \eta w s_{10}\right]^{\alpha}, & i = 1.\end{cases}$$

Plugging the land rents back into the ratios W_i/R_i, we obtain the wages as functions of r, w, and s_{ij}:

$$W_i = \begin{cases}\alpha A_0\left[(1+\eta)s_{00} + 2s_{10} + 2\eta w^{-1}s_{01}\right]^{-(1-\alpha)}, & i = 0;\\ \alpha A_1\left[(1+\phi^2)(1+\eta)s_{11} + s_{01} + \eta w s_{10}\right]^{-(1-\alpha)}, & i = 1.\end{cases}$$

Q.E.D.

B. Proof of Proposition 2

Using (7) and (A.1) – (A.2), we can define the relative demand for land, $\lambda(r,w)$, as follows:

$$\frac{N_0 + H_0}{N_1 + H_1} = \lambda(r,w) \equiv \frac{(1+\eta)d^{\varepsilon}w^{1+\varepsilon}b^{\varepsilon}r^{-\beta\varepsilon} + 2\phi d^{\varepsilon}w^{1+\varepsilon} + 2\eta\phi b^{\varepsilon}r^{-\beta\varepsilon}}{\phi b^{\varepsilon}r^{-\beta\varepsilon} + \eta\phi d^{\varepsilon}w^{1+\varepsilon} + (1+\phi^2)(1+\eta)}\frac{1}{r}. \tag{B.1}$$

Because each location has one unit of land, the relative supply of land is equal to one. In equilibrium, the relative demand for land equals the relative supply of land: $\lambda(r,w) = 1$. Using (B.1) and (6), the condition $\lambda(r,w) = 1$ becomes:

$$\frac{(1+\eta)d^{-1}\omega^{\frac{1+\varepsilon}{\varepsilon}}\rho + 2\phi d^{-1}\omega^{\frac{1+\varepsilon}{\varepsilon}} + 2\eta\phi\rho}{\phi\rho + \eta\phi d^{-1}\omega^{\frac{1+\varepsilon}{\varepsilon}} + (1+\phi^2)(1+\eta)} = b^{1/\beta}\rho^{-\frac{1}{\beta\varepsilon}},$$

whose solution in $\omega^{\frac{1+\varepsilon}{\varepsilon}}$ yields (24).

To derive (25), let us restate (15), using (6) and (23), as follows

$$\left(d^{-1}\omega^{\frac{1}{\varepsilon}}\right)^{\alpha}\left(b^{\frac{1}{\beta}}\rho^{-\frac{1}{\beta\varepsilon}}\right)^{1-\alpha}=a\left(\omega\frac{\rho+2\phi}{1+\phi\rho+\phi^{2}}\right), \tag{B.2}$$

where $a(\cdot)$ is the relative TFP given by (16). Equation (B.2) defines implicitly a function $\omega^{\frac{1+\varepsilon}{\varepsilon}} = g(\rho,\gamma,\delta)$. If $\delta > 0$, then the g-function cannot be expressed in closed form but can be written as a fixed point given by the second line of (25). When $\delta = 0$, the g-function can be expressed in closed form. Indeed, in this case, (B.2) takes the form

$$d^{-\alpha}\left(\omega^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\alpha}{1+\varepsilon}}\left(b^{\frac{1}{\beta}}\rho^{-\frac{1}{\beta\varepsilon}}\right)^{1-\alpha}=c\left(\omega^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\gamma\varepsilon}{1+\varepsilon}}\left(\frac{\rho+2\phi}{1+\phi\rho+\phi^{2}}\right)^{\gamma},$$

whose solution in $\omega^{\frac{1+\varepsilon}{\varepsilon}}$ delivers the first line of (25). Q.E.D.

C. Lemma 1

Lemma 1 *The f-function in the RHS of (24) has the following properties:*

(i) there exist $\rho_m > 0$ and $\rho_M > \rho_m$, such that $f(\rho) > 0$ if and only if $\rho_m < \rho < \rho_M$;

(ii) $f(\rho)$ decreases from ∞ to 0 over (ρ_m,ρ_M).

Proof. The proof follows directly from the properties of the relative demand for land, $\lambda(r,w)$, given by (B.1).

The relative demand for land decreases with the relative land price r. Indeed, computing the elasticity of the relative demand for land w.r.t. r, we get:

$$-\frac{\partial\ln\lambda(r,w)}{\partial\ln r}=$$
$$1+\frac{(1+\eta)\beta\varepsilon\left[\left(1-\phi^{2}+(1+2\phi^{2})\eta\right)d^{\varepsilon}w^{1+\varepsilon}+\eta\phi\left(d^{\varepsilon}w^{1+\varepsilon}\right)^{2}+2\eta\phi(1+\phi^{2})\right]b^{\varepsilon}r^{-\beta\varepsilon}}{r\lambda(r,w)\left[\phi b^{\varepsilon}r^{-\beta\varepsilon}+\eta\phi d^{\varepsilon}w^{1+\varepsilon}+(1+\phi^{2})(1+\eta)\right]^{2}},$$

where the RHS is clearly positive. Also, the relative demand for land increases with the relative wage w. Computing the elasticity of $\lambda(r,w)$ w.r.t. the relative wage w, we get:

$$\frac{\partial\ln\lambda(r,w)}{\partial\ln w}=$$
$$\frac{(1+\eta)(1+\varepsilon)\left[\phi b^{2\varepsilon}r^{-2\beta\varepsilon}+\left(1+3\phi^{2}+\eta(1-\phi^{2})\right)b^{\varepsilon}r^{-\beta\varepsilon}+2\phi(1+\phi^{2})\right]w^{1+\varepsilon}}{r\lambda(r,w)\left[\phi b^{\varepsilon}r^{-\beta\varepsilon}+\eta\phi d^{\varepsilon}w^{1+\varepsilon}+(1+\phi^{2})(1+\eta)\right]^{2}},$$

where the RHS is clearly positive. This result reflects two effects: (i) a higher wage leads to substituting labor with land in production; (ii) the citizens, who are commuting-averse, tend to live in locations offering higher wages.

To derive ρ_m and ρ_M, consider two extreme cases.

Extreme case 1: $w = 0$. The condition $\lambda(r,w) = 1$ becomes:

$$\lambda(r,0) \equiv \frac{2\eta\phi b^{\varepsilon} r^{-\beta\varepsilon}}{\phi b^{\varepsilon} r^{-\beta\varepsilon} + (1+\phi^2)(1+\eta)} \frac{1}{r} = 1. \tag{C.1}$$

The equation $\lambda(r,0) = 1$ has a unique solution $\underline{r} > 0$.

Extreme case 2: $w = \infty$. The condition $\lambda(r,w) = 1$ becomes:

$$\lambda(r,\infty) \equiv \left(\frac{1+\eta}{\eta\phi} b^{\varepsilon} r^{-\beta\varepsilon} + \frac{2}{\eta}\right) \frac{1}{r} = 1. \tag{C.2}$$

The equation $\lambda(r,\infty) = 1$ has a unique solution $\overline{r} > \underline{r} > 0$. That $\overline{r} > \underline{r}$ follows from $\frac{\partial \ln \lambda(r,w)}{\partial \ln w} > 0$, which implies $\lambda(r,\infty) > \lambda(r,0)$ for every given r, hence $\lambda(\overline{r},\infty) = 1 = \lambda(\underline{r},0) < \lambda(\underline{r},\infty)$, which implies $\overline{r} > \underline{r}$.

The above analysis brings us to two important conclusions:

- the equilibrium condition $\lambda(r,w) = 1$ defines an increasing relation between r and w, hence it defines a decreasing relation between ω and ρ;
- while the equilibrium condition $\lambda(r,w) = 1$ can hold for any $w \geq 0$ (including $w = 0$ and $w = +\infty$), it can hold only for a limited range of relative land rents: $r \in [\underline{r},\overline{r}]$.

Because $\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho)$ is just an equivalent way of writing the equilibrium condition $\lambda(r,w) = 1$, which defines a decreasing relationship between ρ and ω, we have $f'(\rho) < 0$ for each admissible value of ρ. Furthermore, because the equilibrium condition $\lambda(r,w) = 1$ can only hold for $r \in [\underline{r},\overline{r}]$, the extreme cases of $r = \underline{r}$ and $r = \overline{r}$, which correspond, respectively, to $w = 0$ and $w = \infty$, the function f decreases from ∞ to 0 as ρ changes from $\rho_m \equiv \left(b\overline{r}^{-\beta}\right)^{\varepsilon}$ to $\rho_M \equiv \left(b\underline{r}^{-\beta}\right)^{\varepsilon} > \rho_m$. Beyond the interval (ρ_m,ρ_M), the expression for $f(\rho)$ in (24), although still mathematically well defined, has no economic meaning. Q.E.D.

D. Lemma 2

Lemma 2 *(i) If $\gamma \neq \gamma_m$ and $\delta = 0$, then $g(\rho;\gamma)$ is strictly positive and finite over $[\rho_m,\rho_M]$. (ii) If $\gamma < \gamma_m$, then g is increasing over $[\rho_m,\rho_M]$. (iii) If $\gamma > \gamma_m$, then g is decreasing over $[\rho_m,\rho_M]$. (iv) As γ converges to γ_m, we have:*

$$\lim_{\gamma \nearrow \gamma_m} g(\rho;\gamma) = \begin{cases} 0, & \rho < \rho_L; \\ \left(\frac{\rho_L+2\phi}{1+\phi\rho_L+\phi^2}\right)^{-\frac{1+\varepsilon}{\varepsilon}} & \rho = \rho_L; \\ \infty, & \rho > \rho_L; \end{cases} \qquad \lim_{\gamma \searrow \gamma_m} g(\rho;\gamma) = \begin{cases} \infty, & \rho < \rho_L; \\ \left(\frac{\rho_L+2\phi}{1+\phi\rho_L+\phi^2}\right)^{-\frac{1+\varepsilon}{\varepsilon}} & \rho = \rho_L; \\ 0, & \rho > \rho_L; \end{cases}$$

where $\rho_L > 0$ is the unique solution to the equation

$$\rho^{\frac{1}{\eta}} \frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} = \left(d\, c^{\frac{1}{\alpha}}\, b^{-\frac{1}{\eta}}\right)^{-\varepsilon}.$$

Proof. Part (i) follows from combining (25) with $0 < \rho_m < \rho_M < \infty$. Parts (ii) and (iii) are obtained by differentiating g with respect to ρ. Part (iv) holds because $g(\rho;\gamma)$ may be rewritten as follows:

$$g(\rho;\gamma) = \Phi \left[\left(\rho^{\frac{1}{\eta}} \frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} \right)^{\frac{\gamma_m}{\gamma_m - \gamma}} \cdot \left(\frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} \right)^{-1} \right]^{\frac{1+\varepsilon}{\varepsilon}}, \qquad \text{(D.1)}$$

where $\Phi \equiv c^{\frac{\psi\eta}{\gamma_m-\gamma}} d^{\frac{\alpha\psi\eta}{\gamma_m-\gamma}} b^{-\frac{\alpha\psi}{\gamma_m-\gamma}}$. Q.E.D.

E. Lemma 3

Lemma 3 *There exists a function $\overline{\phi}(\beta\varepsilon) \in (0,1)$ and scalar $\overline{\eta} > 0$ such that $\rho_m < \rho_L$ if $\phi < \overline{\phi}$ or $\eta < \overline{\eta}$. Conversely, if $\phi > \overline{\phi}$ and $\eta > \overline{\eta}$, then $\rho_m > \rho_L$.*

Proof. It follows from the proof of Lemma 1 that ρ_m is the unique solution of

$$D(\rho) \equiv (1+\eta)\rho^{\frac{1+\beta\varepsilon}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - \eta\phi = 0. \qquad \text{(E.1)}$$

The expressions (D.1) and (E.1) imply that ρ_L and ρ_m are functions of η. We next show that ρ_m and ρ_L vary with η as follows,

$$\lim_{\eta \to 0} \rho_L = 1, \quad \frac{d\rho_L}{d\eta} < 0, \quad \lim_{\eta \to \infty} \rho_L = 1 - \phi,$$

$$\lim_{\eta \to 0} \rho_m = 0, \quad \frac{d\rho_m}{d\eta} > 0, \quad \lim_{\eta \to \infty} \rho_m = \phi^{\frac{\beta\varepsilon}{1+\beta\varepsilon}}.$$

We can show that ρ_m (resp., ρ_L) increases (resp., decreases) in η by applying the implicit function theorem to $D(\rho) = 0$ (resp., (D.1)). Observe further that, when $\eta \to \infty$ (resp., $\eta \to 0$), dividing $D(\rho) = 0$ by η and taking the limit yields $\rho_m = \phi^{\beta\varepsilon/(1+\beta\varepsilon)}$ (resp., $\rho_m = 1$). Last, when $\eta \to \infty$ (resp., $\eta \to 0$), taking (E.1) at the power η and the limit yields $\rho_L = 1 - \phi$ (resp., $\rho_L = 1$).

To determine where ρ_m and ρ_L intersect, we compare $\lim_{\eta\to\infty} \rho_L$ and $\lim_{\eta\to\infty} \rho_m$ by considering the equation

$$\phi^{\beta\varepsilon/(1+\beta\varepsilon)} + \phi = 1. \tag{E.2}$$

Differentiating the LHS of (E.2) with respect to ϕ shows that it increases from 0 to 2 when ϕ increases from 0 to 1. The intermediate value theorem then implies that, for any given $\beta\varepsilon$, (E.2) has a unique solution $\overline{\phi}(\beta\varepsilon) \in (0,1)$, which increases with $\beta\varepsilon$.

The inequality $\rho_m \leq \rho_L$ holds if $\phi^{\beta\varepsilon/(1+\beta\varepsilon)} \leq 1 - \phi$, which amounts to $\phi \leq \overline{\phi}(\beta\varepsilon)$. If $\overline{\phi} < \phi \leq 1$, then there exists a unique value $\overline{\eta} > 0$ that solves the condition $\rho_L(\eta) = \rho_m(\eta)$. Consequently, if $\eta < \overline{\eta}$, then $\rho_m \leq \rho_L$. If $\eta \geq \overline{\eta}$, then $\rho_m > \rho_L$. Summing up, $\rho_m \leq \rho_L$ if $\phi \leq \overline{\phi}$ or $\eta \leq \overline{\eta}$, and $\rho_m > \rho_L$ when both conditions fail. Q.E.D.

F. Proof of Proposition 3

Plugging $b = c = d = 1$ and $\gamma = \delta = 0$ into (24), we get

$$f(\rho) \equiv \frac{\phi\rho - 2\eta\phi\rho^{1+\frac{1}{\beta\varepsilon}} + (1+\phi^2)(1+\eta)}{(1+\eta)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - \eta\phi}, \qquad g(\rho) = \rho^{\psi}.$$

(i) Because $f(\rho)$ decreases with ρ from ∞ to 0 over (ρ_m, ρ_M), and $g(\rho)$ increases with ρ from 0 to ∞, the two curves have a unique intersection $\rho^* \in (\rho_m, \rho_M)$, which implies existence and uniqueness of equilibrium and that it is interior.

(ii) It is readily verified that $0 < f(1) < g(1) = 1$. Hence, the intersection must occur strictly between $\rho_m < 1$ and 1. This implies $0 < \rho^* < 1$ and $\omega^* = (\rho^*)^{\frac{1}{\eta}} < 1$.

(iii) The equilibrium employment pattern is bell-shaped if and only if

$$\ell^* = \omega^* \frac{\rho^* + 2\phi}{1 + \phi\rho^* + \phi^2} = (\rho^*)^{\frac{1}{\eta}} \frac{\rho^* + 2\phi}{1 + \phi\rho^* + \phi^2} > 1.$$

Restate the equilibrium condition $f(\rho) = g(\rho)$ as follows:

$$\frac{\left(\frac{\eta}{1+\eta}\rho^{-\psi} + \frac{1}{1+\eta}\rho^{-1}\right)^{-1} + 2\phi}{\phi\left(\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho\right) + 1 + \phi^2} \left(\frac{\eta}{1+\eta}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\rho^{\psi+\frac{1}{\beta\varepsilon}}\right) = 1. \tag{F.1}$$

Because $1/x$ is convex, for every $\rho < 1$ Jensen's inequality implies

$$\left(\frac{\eta}{1+\eta}\rho^{-\psi} + \frac{1}{1+\eta}\rho^{-1}\right)^{-1} < \frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho < \rho. \tag{F.2}$$

Plugging (F.2) into (F.1) leads to

$$1 < \frac{\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho + 2\phi}{\phi\left(\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho\right) + 1 + \phi^2}\left(\frac{\eta}{1+\eta}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\rho^{\psi+\frac{1}{\beta\varepsilon}}\right).$$

Using $\psi > 1$ yields

$$\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho < \frac{\eta}{1+\eta}\rho + \frac{1}{1+\eta}\rho = \rho.$$

Because the function $\frac{x+2\phi}{\phi x+1+\phi^2}$ is increasing for all $x \geq 0$, we obtain

$$1 < \frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2}\left(\frac{\eta}{1+\eta}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\rho^{\psi+\frac{1}{\beta\varepsilon}}\right). \tag{F.3}$$

As $\psi > 1$ implies

$$\frac{1}{\eta} < 1 + \frac{1}{\beta\varepsilon} < \psi + \frac{1}{\beta\varepsilon},$$

while $\rho^* < 1$, we have

$$\frac{\eta}{1+\eta}\left(\rho^*\right)^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\left(\rho^*\right)^{\psi+\frac{1}{\beta\varepsilon}} < \left(\rho^*\right)^{\frac{1}{\eta}}.$$

Replacing the bracketed term in (F.3), we obtain the inequality:

$$1 < \left(\rho^*\right)^{\frac{1}{\eta}}\frac{\rho^* + 2\phi}{\phi\rho^* + 1 + \phi^2},$$

which is equivalent to $\rho^* > \rho_L$, hence $\ell^* > 1$ (see (7)).

(iv) When $\varepsilon \to \infty$, we have:

$$\lim_{\varepsilon\to\infty} f(\rho) = \rho^{-1}, \; \lim_{\varepsilon\to\infty} g(\rho) = \rho^{\frac{1}{\eta}} \implies \lim_{\varepsilon\to\infty} \rho^* = \lim_{\varepsilon\to\infty} \omega^* = 1.$$

Also, $\lim_{\varepsilon\to\infty}\phi = 0$. Hence, taking the limit w.r.t. $\varepsilon \to \infty$ on both sides of (7), we get:

$$\lim_{\varepsilon\to\infty}\begin{pmatrix} s_{11} & s_{10} & s_{1-1} \\ s_{01} & s_{00} & s_{0-1} \\ s_{-11} & s_{-10} & s_{-1-1} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Q.E.D.

G. Proof of Proposition 4

We first prove the following lemma.

Lemma 4 *Consider an interior equilibrium* (ω^*, ρ^*)*, such that* $g'(\rho^*) > f'(\rho^*)$*. Any shock in* c*,* γ *or* δ *that shifts the* g*-curve upwards/downwards in the vicinity of the equilibrium leads to a labor pattern more/less concentrated at the center.*

Proof. Combining the labor centrality ratio (23) with (24), we get:

$$\ell^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho)\left(\frac{\rho + 2\phi}{1 + \phi\rho + \phi^2}\right)^{\frac{1+\varepsilon}{\varepsilon}}. \tag{G.1}$$

It is readily verified that the RHS of (G.1) decreases in ρ. Because f is independent of c, the RHS of (G.1) as a function of ρ is also independent of c. Note, however, that the equilibrium value of ρ depends on c, which implies that the equilibrium value of ℓ varies with c. Indeed, an upward/downward shift in the g-curve leads to a decrease/increase in ρ^* because the equilibrium moves northwestwards/southeastwards along the f-curve which is unaffected by the change in the value of c.

Hence, we have:

$$\text{an upward shift in } g \Longrightarrow \rho^* \downarrow \Longrightarrow \ell^* \uparrow .$$

Q.E.D.

From (24) – (25), one can see that an increase in c keeps the f-curve unchanged and shifts upwards the g-curve. Hence, by Lemma 4, we have:

$$\frac{d\ell^*}{dc} > 0. \tag{G.2}$$

In other words, ℓ^* is strictly increasing in c.

We now prove (i) – (v) in the following order: (i), (v), (iii), (ii) and (iv).

(i) We need to show that

$$\lim_{c\searrow 0} \ell^* = 0.$$

Because $\lim_{c\searrow 0} g(\rho) = 0$ for $\forall\, \rho \in (\rho_m, \rho_M)$, we have:

$$\begin{cases} \lim\limits_{c\searrow 0} \omega^*(c) = 0 \\ \lim\limits_{c\searrow 0} \rho^*(c) = \rho_M \end{cases} \Longrightarrow \quad \lim_{c\searrow 0} \ell^*(c) = \lim_{c\searrow 0}\left[\frac{\rho^*(c) + 2\phi}{1 + \phi\rho^*(c) + \phi^2}\omega^*(c)\right] = 0. \tag{G.3}$$

(v) Along the same lines as in the proof of (i), one can show that

$$\begin{cases} \lim\limits_{c\nearrow\infty} \omega^*(c) = \infty \\ \lim\limits_{c\nearrow\infty} \rho^*(c) = \rho_m \end{cases} \qquad \Longrightarrow \qquad \lim_{c\nearrow\infty} \ell^*(c) = \lim_{c\nearrow\infty} \left[\frac{\rho^*(c) + 2\phi}{1 + \phi\rho^*(c) + \phi^2} \omega^*(c) \right] = \infty. \tag{G.4}$$

(iii) Using the implicit function theorem shows that $\ell^*(c)$ is differentiable, hence continuous, w.r.t. c for $0 < c < \infty$. Combining this with (G.2) – (G.4) implies that the equation $\ell^*(c) = 1$ has a unique, finite, and positive, solution c_0.

(ii) and (iv) follow from (G.1) combined with (i), (iii), and (v). Q.E.D.

H. Proof of Proposition 5

Our proof strategy is to study whether a slight increase in γ and δ above zero shifts the g-function upward or downward, and then apply Lemma 4. Because the g-curve is defined implicitly by (25) for $\delta > 0$, all we need are the first-order impacts of γ and δ on the relative productivity $a(\ell)$.

(i) Assume that $\delta = 0$. Using (16) implies

$$E(\gamma) = C_0 \left(\frac{\ell}{\ell + 2} \right)^{\gamma}, \qquad F(\gamma) = C_1 \left(\frac{1}{\ell + 2} \right)^{\gamma} \Longrightarrow a(\gamma) = \frac{E(\gamma)}{F(\gamma)} = \frac{C_0}{C_1} \ell^{\gamma}.$$

Therefore,

$$\left. \frac{da(\gamma)}{d\gamma} \right|_{\gamma=0} = c \ln \ell.$$

Proposition 4 implies that $\hat{c} > 0$ exists such that $\ell(\hat{c}) = 1$. Therefore, increasing returns magnifies the initial advantage of a location given by the value of c when $c > \hat{c}$ because raising γ above 0 increases $a(\gamma)$. The opposite holds when $c < \hat{c}$. Furthermore, the intensity of the effect of IRS increases with the value of $c > \hat{c}$.

(ii) Assume that $\gamma = 0$. In this case, we have $a(\ell) = A(\delta)/B(\delta)$ where

$$A(\delta) = C_0 + \frac{2\delta}{\ell + 2}, \qquad B(\delta) = C_1 + \delta \frac{\ell + \delta}{\ell + 2}.$$

Differentiating $A(\delta)/B(\delta)$ with respect to δ and setting $\delta = 0$ yield

$$\left. \frac{da(\delta)}{d\delta} \right|_{\delta=0} > 0 \Longleftrightarrow 4 - 2\ell c + 2\ell - \ell^2 c > 0 \Longleftrightarrow c < \frac{2}{\ell}.$$

Because Proposition 4 implies that $\ell(c)$ is increasing, the equation $2/c - \ell = 0$ has a unique positive solution $\widetilde{c}$. When $c > \widetilde{c}$, stronger spillovers demagnify the initial advantage of the central location because raising δ above 0 decreases $a(\gamma)$. In contrast, when $c < \widetilde{c}$, stronger spillovers increases employment in the central agglomeration of employment. Q.E.D.

I. Proof of Proposition 6

(i) When $\delta = 0$ and $\gamma \neq \gamma_m$,[13] from Lemmas 1 and 2, $f(\rho) > g(\rho,\gamma)$ when ρ slightly exceeds ρ_m, while the opposite inequality holds when ρ is close enough to ρ_M. Hence, by the intermediate value theorem, the equilibrium condition $f(\rho) = g(\rho,\gamma)$ has an interior solution $\rho^* \in (\rho_m,\rho_M)$.

(ii) We now show existence of the two corner equilibria. Consider first the wage pattern $W_0 = 0 < W_1$, hence $\omega = w = 0$. The utility-maximizing commuting flows (7) imply that, at the central location $i = 0$, labor supply $= 0 \Longrightarrow A_0 = 0 \Longrightarrow$ labor demand $= 0$. The land-market clearing condition $\lambda(r,w) = 1$ takes the form of (C.1), which means $r^* = \underline{r} \Longrightarrow \rho^* = \rho_M$.

(iii) Last, we show that $M_i^* > 0$ for all i. Assume that $R_i^* = 0$ at i. Because there is a location j such that $W_j^* > 0$, workers who choose the pair ij enjoys an infinite utility level, which implies $s_{ij} > 0$. These workers' land demand is thus infinite while the land supply is finite, a contradiction. Q.E.D.

J. Proof of Proposition 7

Under weak IRS, given Lemmas 1 and 2, f and g must intersect exactly once. Furthermore, because $f(1) < 1 < g(1;\gamma)$, the intersection must occur at $\rho^* < 1$. Because $\rho^* > \rho_L$, we have

$$f(\rho_L) > f(\rho^*) = g(\rho^*) > g(\rho_L) \tag{J.1}$$

because f is decreasing by Lemma 1 and g is increasing in ρ by Lemma 2. As shown by (D.1), $g(\rho_L)$ is independent of γ. Combining this with (J.1), we obtain $f(\rho_L) - g(\rho_L;\gamma) > 0$. Because $f(\rho^*) - g(\rho^*;\gamma) = 0$ while $f - g$ is decreasing by Lemmas 1 and 2, we have $\rho_L < \rho^*$ for all $\gamma < \alpha/\varepsilon$, which amounts to $\ell^* > 1$.

We now study the impact of γ on (i) ρ^*, (ii) $\omega^*$and (iii) ℓ^*.

(i) Because $\partial g(\rho;\gamma)/\partial\gamma > 0$, applying the implicit function theorem to (26) leads to

$$\frac{d\rho^*}{d\gamma} = \left.\frac{\partial g(\rho;\gamma)/\partial\gamma}{\partial f'(\rho)/\partial\rho - \partial g(\rho;\gamma)/\partial\rho}\right|_{\rho=\rho^*} < 0,$$

[13]the case where $\gamma = \gamma_m$ is discussed in Section 7.2.

where the numerator is positive because $\rho^* > \rho_L$ while the denominator is negative because $f(\rho)$ is decreasing and $g(\rho;\gamma)$ is increasing in ρ.

(ii) Differentiating (24) with respect to γ, we obtain:

$$\frac{1+\varepsilon}{\varepsilon}\omega^{\frac{1}{\varepsilon}}\frac{d\omega^*}{d\gamma} = \frac{df}{d\rho}\frac{d\rho^*}{d\gamma} > 0.$$

(iii) From Lemma 4, $d\ell^*/d\gamma > 0$. Q.E.D.

K. Proof of Proposition 8

Step 1. Consider first the case when the spatial discount factor is small ($\phi < \overline{\phi}$), so that $\rho_m < \rho_L < 1 < \rho_M$ holds. Therefore, for $\Delta > 0$ sufficiently small, we have:

$$\rho_m + \Delta < \rho_L - \Delta < \rho_L + \Delta < 1 < \rho_M.$$

If γ is sufficiently close to α/ε (but still such that $\gamma > \alpha/\varepsilon$ holds), Lemma 2 implies the following inequalities:

$$\begin{aligned} g(\rho_m + \Delta;\gamma) &< f(\rho_m + \Delta), \\ g(\rho_L - \Delta;\gamma) &> f(\rho_L - \Delta), \\ g(\rho_L + \Delta;\gamma) &< f(\rho_L + \Delta), \\ g(\rho_M;\gamma) &> f(\rho_M) = 0, \end{aligned}$$

where the last inequality holds because (25) implies that, for $\gamma > \alpha/\varepsilon$, $g(\rho;\gamma) > 0$ for all $\rho > 0$ while $f(\rho_M) = 0$ for any γ by definition of ρ_M. Therefore, by continuity of f and g, (26) has at least *three* distinct solutions, which we denote as follows:

$$\rho_M > \rho_2^\star > \rho_3^\star.$$

Furthermore, the properties of function g imply the following:

$$\lim_{\gamma\varepsilon\searrow\alpha} \rho_1^\star = \rho_M,$$

$$\lim_{\gamma\varepsilon\searrow\alpha} \rho_2^\star = \rho_L,$$

$$\lim_{\gamma\varepsilon\searrow\alpha} \rho_3^\star = \rho_m.$$

The solution $\rho_2^\star$ matches the equilibrium of Proposition 7. As for the other two solutions, $\rho_1^\star$ and $\rho_3^\star$, when γ is close enough to α/ε, we have $\rho_2^\star > 1 > \rho_3^\star$.

As $\gamma \searrow \alpha/\varepsilon$, it follows from Lemma 1 that $f(\rho_2^\star)$ and $f(\rho_3^\star)$ converge, respectively, to 0 and ∞, which implies:

$$\lim_{\gamma\varepsilon \searrow \alpha} \omega_1^\star = 0 \qquad \text{and} \qquad \lim_{\gamma\varepsilon \searrow \alpha} \omega_3^\star = \infty.$$

Hence, $\omega_1^\star < 1 < \omega_3^\star$ when $\gamma\varepsilon$ is close enough to α. It then follows from (25) that

$$\lim_{\gamma\varepsilon \searrow \alpha} \ell_1^\star = 0 \qquad \text{and} \qquad \lim_{\gamma\varepsilon \searrow \alpha} \ell_3^\star = \infty.$$

Step 2. Consider now the case where the spatial discount factor is high ($\phi > \overline{\phi}$). Then, we know from Lemma 3 that there exists a value $\overline{\eta} \in (0,1)$ such that

$$\rho_L \leq \rho_m < 1 < \rho_M \tag{K.1}$$

is satisfied for $\eta \geq \overline{\eta}$, while $\rho_m < \rho_L < 1 < \rho_M$ holds otherwise. Under (K.1), there is a small $\Delta > 0$ such that the following inequalities hold:

$$g(\rho_M - \Delta; \gamma) < f(\rho_M - \Delta),$$
$$g(\rho_M; \gamma) > f(\rho_M) = 0.$$

while $\rho^\star > 1$ when γ slightly exceeds α/ε.

Furthermore,

$$\lim_{\gamma\varepsilon \searrow \alpha} (\omega_1^*)^{\frac{\varepsilon}{1+\varepsilon}} = f(\rho_M) = 0.$$

Because $\lim_{\gamma\varepsilon \searrow \alpha} \omega_1^\star = 0$, $\omega_1^\star < 1$ when $\gamma\varepsilon$ is sufficiently close to α.

Last, using (7), we have:

$$\lim_{\gamma\varepsilon \searrow \alpha} \ell_1^\star = 0.$$

Q.E.D.

L. Proof of Proposition 9

First, we show the existence and uniqueness of an equilibrium. The equilibrium condition (26) can be restated as follows:

$$\frac{1}{\phi\rho + 1 + \phi^2} \left(\frac{1 + \eta \frac{1+\phi^2 - 2\phi\rho^{1+\frac{1}{\beta\varepsilon}}}{\phi\rho + 1 + \phi^2}}{1 + \eta \frac{\rho - \phi\rho^{-\frac{1}{\beta\varepsilon}}}{\rho + 2\phi}} \right)^\lambda = \rho^\mu \frac{\rho}{\rho + 2\phi}, \tag{L.1}$$

where λ and μ are defined by

$$\lambda \equiv \frac{\gamma\varepsilon - \alpha}{\gamma + \alpha} > 0 \qquad \text{and} \qquad \mu \equiv \frac{\gamma\varepsilon - \alpha - (1-\alpha)(1+\varepsilon)}{\beta\varepsilon(\gamma+\alpha)}.$$

The first term of the LHS of (L.1) decreases in ρ; the second term also decreases because the numerator decreases while the denominator increases in ρ. Therefore, the LHS of (L.1) is a decreasing function of ρ. Furthermore, the RHS of (G.1) increases from 0 to ∞ in ρ when $\mu > 0$. It is readily verified that $\mu > 0$ if and only if

$$\gamma > \frac{1+\varepsilon}{(1-\beta)\varepsilon} - \alpha.$$

Hence, (L.1) has a unique solution ρ^*.

We now show that ℓ^* converges monotonically toward 1 when $\gamma > \gamma_s$ increases. Using (7), we obtain

$$\log \ell^* = -\frac{1}{\gamma\varepsilon/\alpha - 1} \log \left((\rho^*)^{\frac{1}{\eta}} \frac{\rho^* + 2\phi}{\phi\rho^* + 1 + \phi^2} \right). \tag{L.2}$$

Because $\rho^* > \rho_L$ under strong increasing returns, the expression under the log is greater than 1 and thus the RHS of (L.2) is negative. Furthermore, as ρ^* decreases with γ, the RHS of (L.2) increases with γ. In addition, the first of the RHS goes to 0 when γ goes to infinity. Consequently, ℓ^* converges to 1. Q.E.D.

M. Lemma 4

Lemma 5 *For any two distinct location pairs ij and kl such that $s^*_{ij} > 0$, there exists an individual $\nu \in [0,1]$ with $z_{ij}(\nu) \in S_{ij}$ and $z_{kl}(\nu) > 0$ who is indifferent between ij and kl.*

Proof. The assumption $s^*_{ij} > 0$ implies $L^*_j > 0$, hence $W^*_j > 0$. Combining this with (2) and (29) implies that any individual $\nu \in [0,1]$ whose type $\mathbf{z}(\nu)$ satisfies

$$z_{ij}(\nu) = z_{kl}(\nu) \frac{W^*_l}{W^*_j} \frac{\tau_{kl}}{\tau_{ij}} \left(\frac{R^*_i}{R^*_k} \right)^{\beta} \geq z_{od}(\nu) \frac{W^*_d}{W^*_j} \frac{\tau_{od}}{\tau_{ij}} \left(\frac{R^*_o}{R^*_i} \right)^{\beta} \tag{M.1}$$

is indifferent between ij and kl.

Two cases may arise. First, if $s^*_{kl} > 0$, then $L^*_l > 0$ and $W^*_l > 0$. (M.1) thus implies that any individual ν satisfying

$$z_{kl}(\nu) > 0, \quad z_{ij}(\nu) = z_{kl}(\nu) \frac{W^*_l}{W^*_j} \frac{\tau_{kl}}{\tau_{ij}} \left(\frac{R^*_i}{R^*_k} \right)^{\beta}, \quad z_{od}(\nu) = 0$$

is indifferent between ij and kl.

Second, if $s^*_{kl} = 0$, then $L^*_l = 0$ and $W^*_l = 0$. Therefore, (M.1) implies that any individual such that $z_{kl}(\nu) > 0$ and $z_{ij}(\nu) = 0$ for any $ij \neq kl$ is indifferent between ij and kl. Q.E.D.

N. Existence and uniqueness of a conditional equilibrium price system

Step 1. We first show the existence of a unique conditional equilibrium price for a symmetric commuting pattern $\mathbf{s}$ such that either $\mathbf{L}(\mathbf{s}) = (0,1,0)$ or $\mathbf{L}(\mathbf{s}) = (1/2,0,1/2)$, and $M_i(\mathbf{s}) > 0$ for $i = 0, \pm 1$.

We focus on the case of a fully agglomerated labor supply pattern, i.e., such that $L_0(\mathbf{s}) = 1$ and $L_{-1}(\mathbf{s}) = L_1(\mathbf{s}) = 0$ (the proof for the fully dispersed labor supply pattern given by $L_0 = 0$ and $L_{-1}(\mathbf{s}) = L_1(\mathbf{s}) = 1/2$ goes along the same lines). Plugging $L_0 = 1$ and $L_{-1} = L_1 = 0$ into the firm's complementary slackness conditions at $i = 0$, we obtain

$$W_0 = \alpha N_0^{1-\alpha} \qquad \text{and} \qquad R_0 = (1-\alpha) N_0^{-\alpha}, \tag{N.1}$$

so that

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha} N_0. \tag{N.2}$$

Observe that $L_1(\mathbf{s}) = L_{-1}(\mathbf{s}) = 0$ implies $s_{i1} = s_{i,-1} = 0$ for all $i \in \{-1,0,1\}$. Combining this with the land market clearing condition and the market residential demand at $i = 0$, we get:

$$H_0 + N_0 = 1 \qquad \text{and} \qquad H_0 = s_{00} \frac{W_0}{R_0},$$

so that

$$N_0 = 1 - H_0 = 1 - s_{00} \frac{W_0}{R_0}. \tag{N.3}$$

Plugging (N.3) into (N.2), we get a linear equation in W_0/R_0:

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha}\left(1 - s_{00}\frac{W_0}{R_0}\right) \quad \Longrightarrow \quad \frac{\overline{W}_0(\mathbf{s})}{\overline{R}_0(\mathbf{s})} = \frac{\alpha}{1-\alpha+\alpha s_{00}}. \tag{N.4}$$

From (N.3)-(N.4), we get:

$$\overline{N}_0(\mathbf{s}) = \frac{1-\alpha}{1-\alpha+\alpha s_{00}}.$$

Plugging $N_0 = \overline{N}_0(\mathbf{s})$ into the equilibrium condition (N.1) pins down uniquely the conditional equilibrium wage $\overline{W}_0(\mathbf{s})$ and the conditional equilibrium land rent $\overline{R}_0(\mathbf{s})$. As for $\overline{W}_j(\mathbf{s})$ and $\overline{R}_i(\mathbf{s})$ for $i,j = \pm 1$, zero labor supply implies $\overline{W}_j(\mathbf{s}) = 0$ and $\overline{N}_j(\mathbf{s}) = 0$ for $j = \pm 1$. Hence, the land market clearing at the periphery becomes

$$H_i = 1 = s_{i0} \frac{W_0}{R_i} \qquad \text{for} \qquad i = \pm 1,$$

which implies $\overline{R}_i(\mathbf{s}) = s_{i0} \overline{W}_0(\mathbf{s})$ for $i = \pm 1$.

O. Instability of the corner equilibria

Assume that $L_0^* = 1$ (the proof for $L_{-1}^* = L_1^* = 1/2$ goes along the same lines). Consider an individual ν such that, for all $i \in \{-1,0,1\}$, ν's match values satisfy $z_{ij}(\nu) = 0$ for $j = 0, \pm 1$. Clearly, ν is indifferent between working at the center and working at the periphery (in both cases, she enjoys zero utility). Consider a positive-measure set of individuals whose tastes are close to those of ν and whose utility-maximizing choice is $ij = 00$. Relocating them (together with ν) from $ij =$ 00 to $kl = 01$, we have $V_{01}(\nu,\mathbf{s}) > 0$ because $\overline{W}_1(\mathbf{s}) > 0$. Using the complementary slackness condition $\left(\alpha A_j L_j^{\alpha-1} N_j^{1-\alpha} - W_j\right) L_j = 0$, there exists a positive-measure subset of individuals who are strictly better-off working at location $j = 1$. As a result, the corner equilibrium $L_0^* = 1$ is an unstable equilibrium. Q.E.D.

P. Proposition 10

Proposition 10 *There exists a function $\mathbb{F}(\rho)$ independent of γ such that an interior equilibrium ρ^* is stable if and only if $\mathbb{F}(\rho^*) > 1$. This function is continuous over (ρ_m,ρ_{CR}) and over (ρ_{CR},ρ_M), satisfies $\mathbb{F}(\rho_m) = \mathbb{F}(\rho_M) = 0$, and has a vertical asymptote at $\rho = \rho_{CR}$.*

Proof: The proof involves four steps.

Step 1. We first show the existence of a unique conditional equilibrium price for a symmetric commuting $\mathbf{s}$ such that $s_{ij} > 0$ for all i,j when $\alpha > 1/2$.

Because $L_i > 0$ for $i = 0, \pm 1$, the first-order conditions for the production sector yields the equilibrium conditions:

$$W_j = \alpha A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha}, \tag{P.1}$$

$$R_j = (1-\alpha) A_j \left(\frac{L_j}{N_j}\right)^{\alpha}. \tag{P.2}$$

Furthermore, we also know that housing market clearing at location i yields:

$$H_i = \frac{\beta}{R_i} \sum_{j=1}^{n} s_{ij} W_j. \tag{P.3}$$

Plugging (P.1) and (P.2) into (P.3), and using the land market balance condition $N_i + H_i = 1$, we get:

$$H_i = 1 - N_i = \frac{\alpha\beta}{(1-\alpha)A_i} \left(\frac{N_i}{L_i}\right)^{\alpha} \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha},$$

$$(1-\alpha)A_i\,(1-N_i)\left(\frac{N_i}{L_i}\right)^{-\alpha} = \alpha\beta \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha},$$

$$(1-\alpha)A_i \left(\frac{N_i}{L_i}\right)^{-\alpha} = (1-\alpha)A_i L_i \left(\frac{N_i}{L_i}\right)^{1-\alpha} + \alpha\beta \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha}.$$

Because $\mathbf{s}$ is symmetric, this system of equations becomes:

$$(1-\alpha)A_0 \left(\frac{N_0}{L_0}\right)^{-\alpha} = \left[(1-\alpha)L_0 + \alpha\beta s_{00}\right] A_0 \left(\frac{N_0}{L_0}\right)^{1-\alpha} + 2\alpha\beta s_{01} A_1 \left(\frac{N_1}{L_1}\right)^{1-\alpha}$$

$$(1-\alpha)A_1 \left(\frac{N_1}{L_1}\right)^{-\alpha} = \alpha\beta s_{10} A_0 \left(\frac{N_0}{L_0}\right)^{1-\alpha} + \left[(1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})\right] A_1 \left(\frac{N_1}{L_1}\right)^{1-\alpha}$$

Dividing one equation by the other and using $A_i = L_i^{\gamma}$ for $i = 0, \pm 1$, we get:

$$n^{-\alpha}\ell^{\gamma+\alpha} = \frac{\left[(1-\alpha)L_0 + \alpha\beta s_{00}\right] \ell^{\gamma} \left(\frac{n}{\ell}\right)^{1-\alpha} + 2\alpha\beta s_{01}}{\alpha\beta s_{10} \ell^{\gamma} \left(\frac{n}{\ell}\right)^{1-\alpha} + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})} \tag{P.4}$$

Because (P.1) and (P.2) imply

$$n^{-\alpha}\ell^{\gamma+\alpha} = r, \qquad \ell^{\gamma}\left(\frac{n}{\ell}\right)^{1-\alpha} = w, \tag{P.5}$$

we have

$$w^{\alpha} r^{1-\alpha} = \ell^{\gamma} = \left(\frac{L_0}{L_1}\right)^{\gamma}. \tag{P.6}$$

Likewise, combining (P.4) and (P.5), we get:

$$r = \frac{[(1-\alpha)L_0 + \alpha\beta s_{00}]\, w + 2\alpha\beta s_{01}}{\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}. \tag{P.7}$$

A sufficient condition for the system (P.6) – (P.7) to have a unique solution $(\overline{w}(\mathbf{s}),\overline{r}(\mathbf{s}))$ is that the graph of the relationship (P.7) between w and r intersects the downward-sloping curve given by (P.6) from below. The RHS of (P.7) is the ratio of two positive linear increasing functions of w. Because the elasticity of a linear increasing function with a positive intercept never exceeds 1, the elasticity of the RHS of (P.7) w.r.t. w is always larger than -1. Restating (P.6) as

$$r = \ell^{\frac{\gamma}{1-\alpha}} w^{-\frac{\alpha}{1-\alpha}}$$

shows that the elasticity of the RHS of this expression w.r.t. w equals $-\alpha/(1-\alpha)$, which is smaller than -1 when $\alpha > 1/2$.

Step 2. Denote by $\left(\overline{\mathbf{W}}(\mathbf{s}),\overline{\mathbf{R}}(\mathbf{s})\right)$ the equilibrium price vector conditional to an arbitrary commuting pattern $\mathbf{s}$ that belongs to a neighborhood of an interior equilibrium commuting pattern $\mathbf{s}^*$, and let $\overline{w}(\mathbf{s})$ and $\overline{r}(\mathbf{s})$ be the corresponding wage ratio and the land-price ratio:

$$\overline{w}(\mathbf{s}) \equiv \frac{\overline{W}_0(\mathbf{s})}{\overline{W}_1(\mathbf{s})} \quad \text{and} \quad \overline{r}(\mathbf{s}) \equiv \frac{\overline{R}_0(\mathbf{s})}{\overline{R}_1(\mathbf{s})}.$$

Consider the following two types of relocations: $0j \to 1j$ (changing place of residence but not the workplace) and $i0 \to i1$ (changing the workplace but not the place of residence). Observe that, in equilibrium, for each individual ν, we have:

$$\frac{V_{0j}^*(\nu)}{V_{1j}^*(\nu)} = \frac{z_{0j}(\nu)}{z_{1j}(\nu)} \left(\overline{r}(\mathbf{s}^*)\right)^{-\beta}, \tag{P.8}$$

$$\frac{V_{i0}^*(\nu)}{V_{i1}^*(\nu)} = \frac{z_{i0}(\nu)}{z_{i1}(\nu)} \overline{w}(\mathbf{s}^*). \tag{P.9}$$

If the individual ν is indifferent between $0j$ and $1j$ for some $j = \{-1,0,1\}$, switching from $0j$ to $1j$ makes this individual strictly worse off if and only if $\overline{r}(\mathbf{s}^*)$ decreases when a small subset of residents (almost indifferent between $0j$ and $1j$) of measure Δ is moved from 0 to 1, i.e.,

$$\frac{\partial \overline{r}(\mathbf{s}^*)}{\partial s_{1j}} - \frac{\partial \overline{r}(\mathbf{s}^*)}{\partial s_{0j}} < 0 \tag{P.10}$$

because (P.8) and (P.10) imply that $V_{0j}^*(\nu)/V_{1j}^*(\nu)$ increases above 1.

Likewise, using (P.9) if ν is an individual indifferent between $i0$ and $i1$ for some $i = \{-1,0,1\}$, switching from $i0$ to $i1$ makes ν strictly worse off if and only if $\overline{w}(\mathbf{s}^*)$ increases when a small subset of workers (almost indifferent between $i0$ and $i1$) of measure Δ is moved from 0 to 1, i.e.,

$$\frac{\partial \overline{w}(\mathbf{s}^*)}{\partial s_{i1}} - \frac{\partial \overline{w}(\mathbf{s}^*)}{\partial s_{i0}} > 0. \tag{P.11}$$

Step 3. We now show that the land-price ratio $\overline{r}(\mathbf{s}^*)$ always satisfies the equilibrium condition (P.10). Under a relocation of residents from $0j$ to $1j$ (or, equivalently, from $1j$ to $0j$) for $j = 0,1$, the numerator in the RHS of (P.7) decreases pointwise, while the denominator increases pointwise. Therefore, the curve (P.7) shifts downwards in the (w,r)-plane, while the curve (P.6) remains unchanged. Because (P.7) intersects (P.6) from below, this implies a reduction in $\overline{r}(\mathbf{s})$. Hence, (P.10) holds.

Step 4. It remains to check when (P.11) holds. To this end, we study when the relocation of a $\Delta-$measure subset of workers from $i0$ to $i1$ for $i = 0, \pm 1$ leads to an increase in the relative wage $\overline{w}(\mathbf{s})$. As a result, two cases must be distinguished: (i) a relocation of workers from 00 to 01 and (ii) a relocation of workers from 10 to 11.

Taking the log-differential of (P.6) yields:

$$\alpha d \log w + (1-\alpha) d \log r = \gamma (d \log L_0 - d \log L_1). \tag{P.12}$$

Two cases may arise.

(i) Assume that

$$ds_{00} = -\Delta, \qquad ds_{01} = ds_{0,-1} = \Delta/2,$$

$$ds_{ij} = 0 \text{ otherwise}.$$

In this case, (P.12) becomes:

$$\alpha \, d \log w + (1-\alpha) \, d \log r = \gamma \left(\frac{ds_{00}}{L_0} - \frac{ds_{01}}{L_1} \right) = -\gamma \Delta \left(\frac{1}{2L_1} + \frac{1}{L_0} \right)$$

Taking the log-differential of (P.7) yields:

$$d\log r = \frac{d\left[\left((1-\alpha)L_0 + \alpha\beta s_{00}\right)w + 2\alpha\beta s_{01}\right]}{\left[(1-\alpha)L_0 + \alpha\beta s_{00}\right]w + 2\alpha\beta s_{01}} - \frac{d\left[\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})\right]}{\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}. \tag{P.13}$$

Because

$$d\left[\left((1-\alpha)L_0 + \alpha\beta s_{00}\right)w + 2\alpha\beta s_{01}\right] = -\Delta\left(1-\alpha+\alpha\beta\right)w + \alpha\beta\Delta + \left((1-\alpha)L_0 + \alpha\beta s_{00}\right)w d\log w,$$

while

$$d\left[\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})\right] = (1-\alpha)\frac{\Delta}{2} + \alpha\beta s_{10}w d\log w,$$

(P.13) becomes

$$d\log r = \left[\frac{-\left(1-\alpha+\alpha\beta\right)w + \alpha\beta}{\left((1-\alpha)L_0 + \alpha\beta s_{00}\right)w + 2\alpha\beta s_{01}} - \frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}\right]\Delta$$

$$+\left[\frac{\left((1-\alpha)L_0 + \alpha\beta s_{00}\right)w}{\left((1-\alpha)L_0 + \alpha\beta s_{00}\right)w + 2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}\right]d\log w$$

Plugging this expression into (P.12), we get:

$$d\log w = \frac{-\gamma\left(\frac{1}{2L_1} + \frac{1}{L_0}\right) + (1-\alpha)\left[\frac{(1-\alpha+\alpha\beta)w - \alpha\beta}{((1-\alpha)L_0 + \alpha\beta s_{00})w + 2\alpha\beta s_{01}} + \frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}\right]}{\alpha + (1-\alpha)\left[\frac{((1-\alpha)L_0 + \alpha\beta s_{00})w}{((1-\alpha)L_0 + \alpha\beta s_{00})w + 2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}\right]}\Delta.$$

When $\alpha > 1/2$, the denominator in $d\log w$ is always positive because each bracketed term of the denominator is smaller than 1. As a result, the stability condition $d\log w > 0$ holds if the numerator is positive:

$$\frac{\left(1-\alpha+\alpha\beta\right)w - \alpha\beta}{\left((1-\alpha)L_0 + \alpha\beta s_{00}\right)w + 2\alpha\beta s_{01}} + \frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})} > \frac{\gamma}{1-\alpha}\left(\frac{1}{L_0} + \frac{1}{2L_1}\right). \tag{P.14}$$

(ii) We now assume that

$$ds_{11} = -ds_{10} = \Delta/2, \qquad ds_{-10} = -ds_{-1,-1} = -\Delta/2,$$

$$ds_{ij} = 0 \text{ otherwise.}$$

Hence, (P.12) becomes:

$$\alpha d\log w + (1-\alpha)\, d\log r = \gamma\left[\frac{ds_{10}+ds_{-10}}{L_0} - \frac{ds_{11}}{L_1}\right] = -\gamma\Delta\left(\frac{1}{2L_1}+\frac{1}{L_0}\right)$$

Because

$$d\left[\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w + 2\alpha\beta s_{01}\right] = -\Delta\,(1-\alpha)\,w + \left((1-\alpha)L_0+\alpha\beta s_{00}\right) wd\log w$$

and

$$d\left[\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11}+s_{1,-1})\right] = \alpha\beta\frac{\Delta}{2}w + (1-\alpha)\frac{\Delta}{2} + \alpha\beta s_{10}wd\log w,$$

(P.13) becomes

$$d\log r = \left[\frac{-\,(1-\alpha)\,w}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}} - \frac{1}{2}\frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]\Delta$$

$$+\left[\frac{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right] d\log w.$$

Plugging this expression for $d\log r$ into (P.12), we get:

$$d\log w \;=\; \frac{-\gamma\left(\frac{1}{2L_1}+\frac{1}{L_0}\right) + (1-\alpha)\left[\frac{(1-\alpha)w}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}} + \frac{1}{2}\frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}{\alpha + (1-\alpha)\left[\frac{((1-\alpha)L_0+\alpha\beta s_{00})w}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}\Delta.$$

If $\alpha > 1/2$, the denominator in $d\log w$ is always positive. Hence, the stability condition $d\log w > 0$ becomes:

$$\frac{(1-\alpha)\,w}{((1-\alpha)L_0+\alpha\beta s_{00})\,w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}>\frac{\gamma}{1-\alpha}\left(\frac{1}{L_0}+\frac{1}{2L_1}\right). \tag{P.15}$$

When $\alpha > 1/2$, the inequalities (P.14) and (P.15) are necessary and sufficient for an interior equilibrium to be stable.

We now rewrite these two conditions in terms of the variable ρ only. Using Proposition 1 and the equilibrium relationship $\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho)$, as well as $\rho = r^{-\beta\varepsilon}$, $\omega = w^{\varepsilon}$, and $\eta = \alpha\beta/(1-\alpha)$, (P.14) and (P.15) become

$$\frac{f(\rho)+\frac{1}{2}(1+\eta)\rho^{-\frac{1}{\beta\varepsilon}}\,[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{((1+\eta)\rho+2\phi)\,f(\rho)+2\eta\phi\rho}>\frac{\gamma}{1-\alpha}\left(\frac{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{2(\phi\rho+1+\phi^2)}+\frac{1}{\rho+2\phi}\right), \tag{P.16}$$

$$\frac{(1+\eta)\,f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}-\eta\right)[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{((1+\eta)\rho+2\phi)\,f(\rho)+2\eta\phi\rho}>\frac{\gamma}{1-\alpha}\left(\frac{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{2(\phi\rho+1+\phi^2)}+\frac{1}{\rho+2\phi}\right), \tag{P.17}$$

Solving the equilibrium condition $f(\rho) = g(\rho;\gamma)$ w.r.t. γ yields

$$\gamma=\frac{\alpha}{1+\varepsilon}\frac{\log\left(\rho^{-\psi}f(\rho)\right)}{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\,[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\right)}.$$

Plugging this expression into (P.16) – (P.17), we get:

$$\begin{aligned}\Phi(\rho)\equiv{}&\frac{2(1-\alpha)(1+\varepsilon)(\phi\rho+1+\phi^2)}{((1+\eta)\rho+2\phi)\,f(\rho)+2\eta\phi\rho}\\&\times\frac{f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}+\frac{\eta}{2}\rho^{-\frac{1}{\beta\varepsilon}}\right)[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\,(\rho+2\phi)+2(\phi\rho+1+\phi^2)}\\&\times\frac{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\,[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\right)}{\alpha\log\left(\rho^{-\psi}f(\rho)\right)}>1,\end{aligned}$$

$$\Psi(\rho) \equiv \frac{2(1-\alpha)(1+\varepsilon)(\phi\rho+1+\phi^2)}{((1+\eta)\rho+2\phi)\, f(\rho)+2\eta\phi\rho} \times \frac{(1+\eta)\, f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}-\eta\right)[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\,(\rho+2\phi)+2(\phi\rho+1+\phi^2)} \times \frac{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\right)}{\alpha\log\left(\rho^{-\psi} f(\rho)\right)} > 1.$$

Last, we set:

$$\mathbb{F}(\rho) \equiv \min\left\{\Phi(\rho),\Psi(\rho)\right\},$$

which is independent of ρ. Verifying $\mathbb{F}(\rho) > 1$ can be done numerically for any vector of parameters by plotting $\mathbb{F}(\rho)$ as a function of the variable ρ. Q.E.D.