Foundations of Cities*

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Abstract: *How do people choose work and residence locations when commuting is costly and productivity spillovers, increasing returns to scale, or first nature advantage, reward the concentration of employment. We describe such an equilibrium city in a simple geography populated by agents with heterogenous preferences over workplace-residence pairs. The behavior of equilibrium cities is more complex than previously understood. Heterogeneous location preferences are sufficient for equilibrium centralization of employment and residence. Increasing returns and productivity spillovers can disperse employment. An increase in commuting costs may decentralize residence and employment. Our results shed new light on classical urban economics and are important for our understanding of quantitative spatial models.*

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1 Introduction

Much of the literature in urban economics can be divided into two categories, classical urban economics and quantitative spatial models (QSM). The classical literature assumes that households have the same preferences; space is continuous and uniform, whether on a line or in a plane; and equilibrium cities are symmetric around a single exogenously selected point. The most influential model in this literature is the monocentric city model. The QSM literature builds on applications of discrete choice models to spatial settings (Anderson *et al.*, 1992; Ben-Akiva and Lerman, 1985). Households have heterogenous preferences over a discrete set of workplace-residence pairs. Model geographies are discrete and empirically founded. Locations are heterogenous in their amenities and productivity, allowing a flexible description of the first nature advantages of locations for work and residence (Redding and Rossi-Hansberg, 2017). Where the classical literature focusses on analytical solutions and qualitative results, QSM concentrates on the numerical evaluation of particular comparative statics in models that describe specific real-world locations.

The advantage of quantitative spatial models is that, unlike classical models, they are flexible enough to be a basis for empirical investigations of actual cities. But such exercises are not purely empirical. Quantitative spatial models are complicated general equilibrium models, and it is rarely clear whether the comparative statics that the usual object of QSM are, like "theorems" that teach us about the model, or if they reflect details of geography to which the model is applied. By providing a nearly complete characterization of a QSM type model, albeit in a stylized geography, we hope provide a basis for thinking about whether the quantitative conclusions of more complicated models are theoretically ambiguous or if they are necessary implications of modelling assumptions.

As in the QSM literature, a single parameter describes household heterogeneity in our model. By varying this parameter we investigate what happens as the heterogenous households approach the homogeneity assumption of the classical literature. In this way, we unify the two literatures. By highlighting the sensitivity of equilibrium to preference heterogeneity, our results invite questions about how and to what extent the equilibrium depends on the distribution of heterogeneous preferences.

We consider a simple setting with three locations where each location is endowed with one unit of land, the simplest geography activities can concentrate in a land-scarce center or disperse to a land-abundant periphery. This geography is rich enough to exhibit previously unremarked comparative statics and to refine our intuition about how economic forces operate to form cities. As is always the case with models that have a small number of locations (e.g., Krugman (1991) or most trade models), our setting precludes immediate empirical application and may rule out even more complex phenomena. However, our approach has the advantages of tractability and transparency, and much of the intuition that we derive appears to be general.

Our main results may be summarized as follows. First, we provide a complete characterization of spatial equilibria in much of the parameter space. By contrast, the existing literature largely restricts attention to parts of the parameter space where equilibrium is unique.

Second, given non-zero commuting costs, we find that preference heterogeneity leads to an 'average preference for central employment and residence'. The intuition behind such average preferences can be seen in a Ricardian trade model. When a location must trade with every other location, the central location has an advantage as the place where average transportation costs are lowest. In our framework, the average preference for centrality arises because an average household commutes everywhere with positive probability. Therefore, preference heterogeneity has an unsuspected implication: even without first nature technological advantage, equilibrium employment may be concentrated at the city center under constant returns. In other words, the city is monocentric. Conversely, when households are homogeneous, regardless of the intensity of increasing returns, there always exists an equilibrium where employment and residence are exactly equal in the center and periphery.

Third, we investigate the relationship between first nature productivity, returns to scale within a location, and productivity spillovers across locations, and the equilibrium arrangement of residence and employment. While all are regarded as mechanisms encouraging the agglomeration of economic activity, we find that equilibrium comparative statics depend sensitively on which of these three mechanisms is at work, and how strongly. For example, under weak increasing returns, there is a unique interior equilibrium, economic activity is centralized, and stronger returns to scale increase central employment, wages and land rents. When increasing returns to scale are moderate, multiple stable interior equilibria may occur and different cities may coexist under identical technological and economic conditions. When returns to scale become even stronger, they lead to dispersed employment, wage equalization, and land rent equalization across locations. That is, returns to scale are not an agglomeration force when they are sufficiently strong.

Fourth, changes in preference heterogeneity also have complex effects on equilibrium. Equilibrium requires that employment and residence be dispersed when households are homogeneous or very heterogeneous. Conversely, concentration of employment and residence occurs in equilibrium when the degree of heterogeneity takes intermediate values.

We also find that the equilibrium employment and residence are dispersed both when commuting costs are high and when they are low. Only at intermediate levels of commuting costs can highly concentrated employment and residence arise. That is, when the location of production and residence is endogenous, the standard intuition that lowering commuting costs leads to dispersed economic activity need not apply.

Finally, it is well known that models with increasing returns or externalities, like ours, often exhibit multiple equilibria (see, e.g., Matsuyama, 1991). In this case, stability is commonly used as an equilibrium refinement to shrink the set of equilibria. We find that standard, iterative methods pervasive in the QSM literature rely on a stability condition that need not be robust to alternative (but equivalent!) formulations of the equilibrium conditions. This problem motivates a new approach to stability. Importantly, we show that multiple stable equilibria may arise in empirically relevant parts of the parameter space. We also provide an algorithm that allows us to determine whether any particular equilibrium is stable. Finally, we show that the corner equilibria, which are pervasive under increasing returns, are unstable.

2 Literature

Most papers in classical urban economics assume households are homogenous, or there are a finite number of classes; and space is continuous and featureless, whether a line or a plane. The simplest, and most influential model in this literature, is the monocentric city model (Fujita, 1989).

Although this workhorse model is otherwise quite general, it relies heavily on the assumption that households choose only their residential location, the location of work being fixed exogenously at the center, and endogenizing the choice of work location has long been an objective of urban economic theory. Ogawa and Fujita (1980) consider a simple setting where firms choose only their location and households choose only their places of work and residence. They introduce the idea of a linear "potential function" as a reduced form description of productivity enhancing spillovers across firms. This assumption, now conventional, requires that firm productivity at any given location responds to a distance weighted mean of employment at all locations. Exploiting spillovers is the source of an agglomeration force, while land scarcity acts as a dispersion force. As the benefits of spillovers increases relative to the cost of commuting, there is first a mixed city, then a duocentric, and finally a monocentric city. This is the foundation for the idea that productivity spillovers are an agglomeration force.

Fujita and Ogawa (1982) build on this initial paper by considering a potential function with negative exponential decay. Unfortunately, even this simple seeming model is difficult to work with. Lucas and Rossi-Hansberg (2002) revisits the problem posed by Fujita and Ogawa (1982), but allow firms and households to substitute between labor and land. They establish general existence and uniqueness results when increasing returns are 'weak enough', but otherwise rely on numerical methods.[1] Using a totally different specification for spillovers, Berliant *et al.* (2002) find that the equilibrium city is always monocentric and involves a specialized CBD.

The fundamentals of QSMs are different from classical urban economics (Redding and Rossi-Hansberg, 2017). First, cities consist of discrete sets of locations connected by a transportation network. Second, by assuming that agents are heterogenous rather than homogenous, QSMs draw on a long history of scholarship that applies discrete choice models to transportation, location and trade problems (Anas, 1983; de Palma *et al.*, 1985; Eaton and Kortum, 2002).

Much of the recent work closely follows Ahlfeldt *et al.* (2015). In this model,

[1] Dong and Ross (2015) cast doubt on the numerical results obtained by Lucas and Rossi-Hansberg and confirms those by Fujita and Ogawa. In particular, in the spirit of Ogawa and Fujita (1980), they find (i) a monocentric pattern with a specialized CBD for low commuting costs, (ii) mixed land use involving firms and workers around the center for moderate commuting costs and (iii) a mixed city for high commuting costs.

households have preferences over housing and consumption, as in the older urban economics literature, and commute from home to work. Households have heterogenous preferences over work-residence pairs and each household selects a unique pair. Locations are heterogenous in their amenities and first nature productivity, while local returns to scale or productivity spillovers are sometimes considered. As a consequence of this realism, analytical results are limited, although it is possible to derive analytic expressions characterizing equilibrium. There are also well-known existence results, and uniqueness has been established for the case when increasing returns are small enough. Beyond this, most of what is known about these models results from the numerical evaluation of particular, empirically founded, comparative statics.

Our analysis is based on a hybrid of the models considered in classical urban and QSM literatures. We use the new quantitative spatial model toolbox to analyze the problem of spatial equilibrium in a stylized geography of the older urban economics literature. As household heterogeneity is parameterized by a single variable, we are able to investigate what happens when the heterogenous households of the QSM literature approach the homogeneity of the older urban economics literature. As we will see, this will lead us to a better understanding of both classes of models and their differences. Our hope is that our study of simple settings like ours will help to illuminate the economic forces at work behind QSM-based counterfactuals and allow a more critical evaluation of their plausibility.

3 Model, equilibrium, and solution method

Model

A city consists of three locations $i,j = -1,0,1$. Each location is endowed with one unit of land. We define a spatial pattern $\mathbf{X} = (X_{-1},X_0,X_1)$ as a triple that specifies the value of X at each location i. We focus on *symmetric* spatial patterns where $X_{-1} = X_1$ and consider only symmetric equilibria. Asymmetric equilibria are not ruled out, but we do not investigate them.

The city is populated by a continuum $[0,1]$ of households and by a competitive production sector whose size is endogenous. All households choose a residence i, a workplace j, their consumption of housing, and a tradable numéraire good. Our model is static and all choices occur simultaneously.

Households have heterogenous preferences over workplace-residence pairs, and household types parameterize preferences. Each household $\nu \in [0,1]$ has a type $\mathbf{z}(\nu) \equiv (z_{ij}(\nu)) \in \mathbb{R}_+^{3\times3}$, a vector of non-negative real numbers, one for each possible workplace-residence pair ij. Following the QSM literature, we assume that the mapping $\mathbf{z}(\nu) : [0,1] \to \mathbb{R}_+^{3\times3}$ is such that the distribution of types is the product measure of 9 identical Fréchet distributions:

$$F(\mathbf{z}) \equiv \exp\left(-\sum_i \sum_j z_{ij}^{-\varepsilon}\right), \tag{1}$$

where $\varepsilon \in (0,\infty)$ describes the heterogeneity of preferences. An increase in ε reduces preference heterogeneity and conversely.

Households commute between workplace and residence. Commuting from i to j involves an iceberg cost $\tau_{ij} \geq 1$. This cost is the same for all households and $\tau_{ij} = 1$ if and only if $i = j$. By symmetry, we assume that the iceberg commuting cost matrix is

$$\begin{pmatrix} \tau_{-1,-1} & \tau_{-1,0} & \tau_{-1,1} \\ \tau_{0,-1} & \tau_{0,0} & \tau_{0,1} \\ \tau_{1,-1} & \tau_{1,0} & \tau_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & \tau & \tau^2 \\ \tau & 1 & \tau \\ \tau^2 & \tau & 1 \end{pmatrix}, \tag{2}$$

where $\tau > 1$.

A household that lives at i and works at j has a Cobb-Douglas indirect utility

$$V_{ij}(\nu) = z_{ij}(\nu)v_{ij}, \quad \text{where } v_{ij} \equiv \frac{W_j}{\tau_{ij} R_i^{\beta}}, \tag{3}$$

where W_j is the wage paid at location j and R_i the land rent at i. Wages are the only source of income because land rent accrues to absentee landlords while perfect competition and constant returns guarantee that equilibrium profits are zero. Because our main purpose is to study the relationship between the production sector technology and spatial equilibrium, to lighten notation, we assume that consumption amenities are the same at each location i and each workplace j.

By solving equation (3) for z_{ij} and substituting in equation (1), we see that the distribution of $V_{ij}(\nu)$ is also Fréchet. Utility maximization requires that households choose the largest of the nine possible work-place residence payoffs that follow from their particular taste parameters z_{ij}. Therefore, using a well-known property of Fréchet measures, it follows that the share s_{ij} of households who choose the location pair ij equals[2]

$$s_{ij} = \frac{v_{ij}^{\varepsilon}}{\sum_{r\in\mathcal{I}}\sum_{s\in\mathcal{I}} v_{rs}^{\varepsilon}} = \frac{\left[W_j/(\tau_{ij}R_i^{\beta})\right]^{\varepsilon}}{\sum_{r\in\mathcal{I}}\sum_{s\in\mathcal{I}}\left[W_s/(\tau_{rs}R_r^{\beta})\right]^{\varepsilon}}. \tag{4}$$

Households that share the same type, ν, choose the same location pair ij and reach the same equilibrium utility level. However, households making the same choice may have different types and different equilibrium utility levels.

Commute cost τ and preference heterogeneity ε almost always occur together as τ^{ε}. To simplify our notation, we introduce the *spatial discount factor* defined by

$$\phi \equiv \tau^{-\varepsilon}, \tag{5}$$

where $\phi \in (0,1)$ decreases with the level of commuting costs and increases with the heterogeneity of the population. Hence, ϕ may be high because either commuting costs are low, or the population is very heterogeneous, or both. It is easy to see that $\phi = 1$ when $\tau = 1$ or $\varepsilon = 0$, while $\phi = 0$ when $\tau \to \infty$ or $\varepsilon \to \infty$. Because the heterogeneity of preferences disappears when $\varepsilon \to \infty$, examining behavior as we approach this limit allows us to examine the role of preference heterogeneity in the model.

Throughout our analysis, we find that equilibrium behavior depends on and can be described by the ratio of central to peripheral quantities. We denote such ratios by lower case letters, e.g., $w \equiv W_0/W_1$ and $r \equiv R_0/R_1$. Given (4), we also define *relative land rents* and *relative wages* as follows:

$$\rho \equiv r^{-\varepsilon\beta}, \quad \omega \equiv w^{\epsilon}. \tag{6}$$

When $\rho > 1$, central land is cheaper than peripheral. When $\omega > 1$ central wages are larger than the periphery.

Using symmetry ($s_{0,1} = s_{0,-1}$, $s_{1,0} = s_{-1,0}$, $s_{1,1} = s_{-1,-1}$ and $s_{1,-1} = s_{-1,1}$) and (2) – (6), Appendix A shows that the commuting flows (4) can be restated as

[2]See, e.g., Eaton and Kortum (2002) equation (8) for a derivation. Anderson *et al.* (1992), Theorem 2.2. gives a more discursive derivation of choice probabilities under the closely related Gumbel distribution.

functions of ρ and ω only:

$$\begin{pmatrix} s_{11} & s_{10} & s_{1-1} \\ s_{01} & s_{00} & s_{0-1} \\ s_{-11} & s_{-10} & s_{-1-1} \end{pmatrix} = \frac{1}{\rho(\omega+2\phi)+2(\phi\omega+1+\phi^2)} \begin{pmatrix} 1 & \phi\omega & \phi^2 \\ \phi\rho & \rho\omega & \phi\rho \\ \phi^2 & \phi\omega & 1 \end{pmatrix}. \quad (7)$$

Equation (7) shows how the heterogeneity of households affect the commuting flows through the value of ϕ. Note that equation (7) follows directly from utility maximization.

Commuting flows determine the residential and employment patterns. Let M_i and L_j be the mass of residents and households at $i,j = 0,1$. Then, we have:

$$\begin{aligned} M_0 &= s_{00} + 2s_{01}, \quad M_1 = s_{10} + (1+\phi^2)s_{11}, \\ L_0 &= s_{00} + 2s_{10}, \quad L_1 = s_{01} + (1+\phi^2)s_{11}. \end{aligned} \quad (8)$$

The labor market clearing and population balance conditions are,

$$L_0 + 2L_1 = M_0 + 2M_1 = 1. \quad (9)$$

Using equations (7) – (8) we can write these conditions in terms of pairwise commute shares.

Let H_i be the amount of residential land and N_i the amount of commercial land at location i. Because each location i is endowed with one unit of land, land market[3] clearing also requires

$$H_i + N_i = 1. \quad (10)$$

The variables that describe our model city are: *residence* $\mathbf{M} \equiv (M_0,M_1)$; *employment* $\mathbf{L} \equiv (L_0,L_1)$; *residential land* $\mathbf{H} \equiv (H_0,H_1)$; *commercial land* $\mathbf{N} \equiv (N_0,N_1)$; the *wages* $\mathbf{W} \equiv (W_0,W_1)$; and *land rent* $\mathbf{R} \equiv (R_0,R_1)$. Recalling our convention of denoting centrality ratios with lower case letters, we have $m \equiv M_0/M_1$, $\ell \equiv L_0/L_1$, $h \equiv H_0/H_1$, and $n \equiv N_0/N_1$. These are the ratios of central to peripheral quantities of residents, employment, residential land, and commercial land.

Assume that the numéraire is produced under perfect competition and the production functions at locations $j = 0,1$ are, respectively,

[3]Note that we implicitly assume an interior solution. To allow for corner equilibria, this condition should be written as $(H_i + N_i - 1)R_i = 0$.

$$Y_0 = A_0 L_0^\alpha N_0^{1-\alpha}, \qquad Y_1 = A_1 L_1^\alpha N_1^{1-\alpha}, \tag{11}$$

where A_j is location-specific TFP and the labor share of output, α, is strictly between zero and one.

Our expressions for TFP A_i provide a nested description of three economic forces conventionally regarded as foundations for the agglomeration of economic activity; first nature technological advantage (C_i), local increasing returns to scale (γ), and productivity spillovers (δ). We assume that increasing returns are localized and that spillovers obey a negative exponential function across locations:

$$A_0 = C_0(L_0^\gamma + 2\delta L_1^\gamma), \qquad A_1 = C_1(L_1^\gamma + \delta L_0^\gamma + \delta^2 L_1^\gamma). \tag{12}$$

We are most often concerned with relative first nature productivity, $c \equiv C_0/C_1$, rather that levels. If the center and periphery have different first nature advantages, production is constant returns, and there are no spillovers, then $\gamma = \delta = 0$ and $c \neq 1$. If there are local increasing returns, but no spillovers and no first nature advantages, then $\gamma > 0$, $\delta = 0$ and $c = 1$. Finally, when $\gamma = 0$, $c = 1$ and $\delta > 0$, then spillovers affect productivity, but there is no local increasing returns or first nature advantage. Thus, our description of TFP permits an investigation of how the organization of a city changes with the intensity and the mechanism that fosters the concentration of employment.

If we set $\delta = 0$, equation (12) describes the technology used by Ciccone and Hall (1996), Duranton and Puga (2004), and Allen and Arkolakis (2014). On the other hand, if $\delta > 0$, our definition of TFP mirrors Fujita and Ogawa (1982) where one unit of employment at one unit distance x contributes $\delta = e^{-x}$ to TFP, and at two units of distance, contributes $\delta^2 = e^{-2x}$ (this suggests 1 as the upper bound of δ).

Our description of TFP differs from Lucas and Rossi-Hansberg (2002) and Ahlfeldt *et al.* (2015) who assume that $A_0 = C_0(L_0 + 2\delta L_1)^\gamma$ and $A_1 = C_1(L_1 + \delta L_0 + \delta^2 L_1)^\gamma$. This specification is inconvenient for our purpose because the impact of a change in spillovers varies with the strength of returns to scale. When there are no spillovers, $\delta = 0$, the two formulations are identical.

If location j hosts a positive share of the production sector, the first-order conditions for cost minimization require $W_j = \alpha A_j \left(N_j/L_j\right)^{1-\alpha}$ and $R_j = (1 -$

$\alpha)\alpha A_j\,(L_j/N_j)^{\alpha}$. Hence, the relative demand for factors is given by

$$\frac{W_j}{R_j} = \frac{\alpha}{1-\alpha}\frac{N_j}{L_j}.$$

Dividing the relative demand at $i = 0$ by the relative demand at $i = 1$, we get:

$$\frac{r}{w} = \frac{\ell}{n}. \tag{13}$$

To satisfy the zero-profit condition, unit cost must equal the price of the numéraire, i.e.,

$$\frac{1}{A_i}\left(\frac{W_i}{\alpha}\right)^{\alpha}\left(\frac{R_i}{1-\alpha}\right)^{1-\alpha} = 1. \tag{14}$$

Dividing (14) at $i = 0$ by the corresponding condition at $i = 1$ yields

$$\frac{w^{\alpha}r^{1-\alpha}}{a(\ell)} = 1, \tag{15}$$

where $a(\ell)$ is the ratio of central and peripheral TFP defined as follows:

$$a(\ell) \equiv \frac{A_0}{A_1} = c\frac{\ell^{\gamma}+2\delta}{\delta\ell^{\gamma}+1+\delta^2}, \tag{16}$$

which increases with ℓ for any $\gamma > 0$ and any $\delta \in (0{,}1)$.

Equilibrium

We now define a spatial equilibrium,

Definition 1. *A spatial equilibrium is a vector of patterns for residence, employment, residential and commercial land, wages and rents* (**M**,**L**,**H**,**N**,**W**,**R**) *such that:*

(i) all households make utility-maximizing choices of workplace, residence, housing, and consumption;

(ii) the production sector minimizes cost in all locations;

(iii) production sector makes zero profit in all locations; and

(iv) all markets at each location clear.

We say a spatial equilibrium is *interior* when each location hosts a positive mass of residents ($M_0 > 0$ and $M_1 > 0$) and produces the consumption good ($Y_0 > 0$ and $Y_1 > 0$). When one or more of these variables is zero, we have a *corner* equilibrium.

The next result shows that in equilibrium, employment, commercial land, and residential land patterns can all be expressed solely in terms of relative rents, ρ and relative wages, ω.

Lemma 1. *In equilibrium, the relative land rents ρ and relative wages ω uniquely determine the residence* **M**, *employment* **L**, *residential land* **H**, and *commercial land* **N** *according to*

$$M_0 = \frac{\rho(\omega + 2\phi)}{\rho(\omega + 2\phi) + 2(\phi\omega + 1 + \phi^2)}, \qquad M_1 = \frac{1 - M_0}{2}, \tag{17}$$

$$L_0 = \frac{\omega(\rho + 2\phi)}{\omega(\rho + 2\phi) + 2(\phi\rho + 1 + \phi^2)}, \qquad L_1 = \frac{1 - L_0}{2}, \tag{18}$$

$$N_0 = \frac{\rho + 2\phi}{\rho + 2\phi + \eta\rho\left(1 + 2\phi\omega^{-\frac{1+\varepsilon}{\varepsilon}}\right)}, \qquad H_0 = 1 - N_0, \tag{19}$$

$$N_1 = \frac{1 + \phi\rho + \phi^2}{1 + \phi\rho + \phi^2 + \eta\left(\phi\omega^{\frac{1+\varepsilon}{\varepsilon}} + 1 + \phi^2\right)}, \qquad H_1 = 1 - N_1. \tag{20}$$

Proof: See Appendix A.

This lemma requires four comments. First, when the relative land rents, ρ, and relative wages, ω, are determined, all the other variables are determined. Second, each location always hosts a positive mass of residents. Indeed, $M_0 < 1$ while $M_0 = 0$ implies $\rho = 0$, that is, $R_0 \to \infty$, which cannot hold in equilibrium. Third, the proof of Lemma 1 shows that the expressions for L (see equation (18)) follow immediately from utility maximization, while the expressions for H and N (see equations (19) and (20)) require market clearing. Fourth, using Lemma 1, the labor supply ratio, ℓ, is equal to

$$\ell(\rho,\omega) \equiv \frac{L_0}{L_1} = \omega\frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2}. \tag{21}$$

As expected, the equilibrium labor supply at the central location increases with the relative wage, w, and decreases with the relative land rent, r. Recalling that $\omega \equiv w^\varepsilon$, it follows immediately from (21) that the heterogeneity parameter ε is the elasticity of the labor supply ratio ℓ with respect to the wage ratio w. Furthermore, for given ρ and ω, the impact of ϕ on $\ell(\rho,\omega)$ is ambiguous.

Solution method

We now turn to a characterization of the spatial equilibrium. To begin, define

$$\eta \equiv \frac{\alpha\beta}{1-\alpha} > 0, \tag{22}$$

Recall that β is the consumption share of housing, and α the production share of labor. Thus, the numerator is the share of firm revenue used for residential land, and the denominator is the share of firm revenue used for commercial land. It follows that η measures the relative intensity of residential and commercial land demand in production. This ratio plays an important role in determining whether production or residence is more concentrated in the land-scarce center. For future reference, we can easily guess at the magnitude of η. A housing share of consumption of $\beta = 0.25$ is in line with modern empirical evidence (Davis and Ortalo-Mangé, 2011). The labor share of output is probably about $\alpha = 0.6$. Substituting into equation (22) we calculate that η is about 0.375.

We use Lemma 1 to write the equilibrium conditions (13) and (15) in terms of ρ and ω. Then, if we solve the two resulting equations for ω and equate them, we are left with a single equation in ρ that is sufficient to determine the interior equilibria. The following proposition provides the foundation for stating this result precisely.

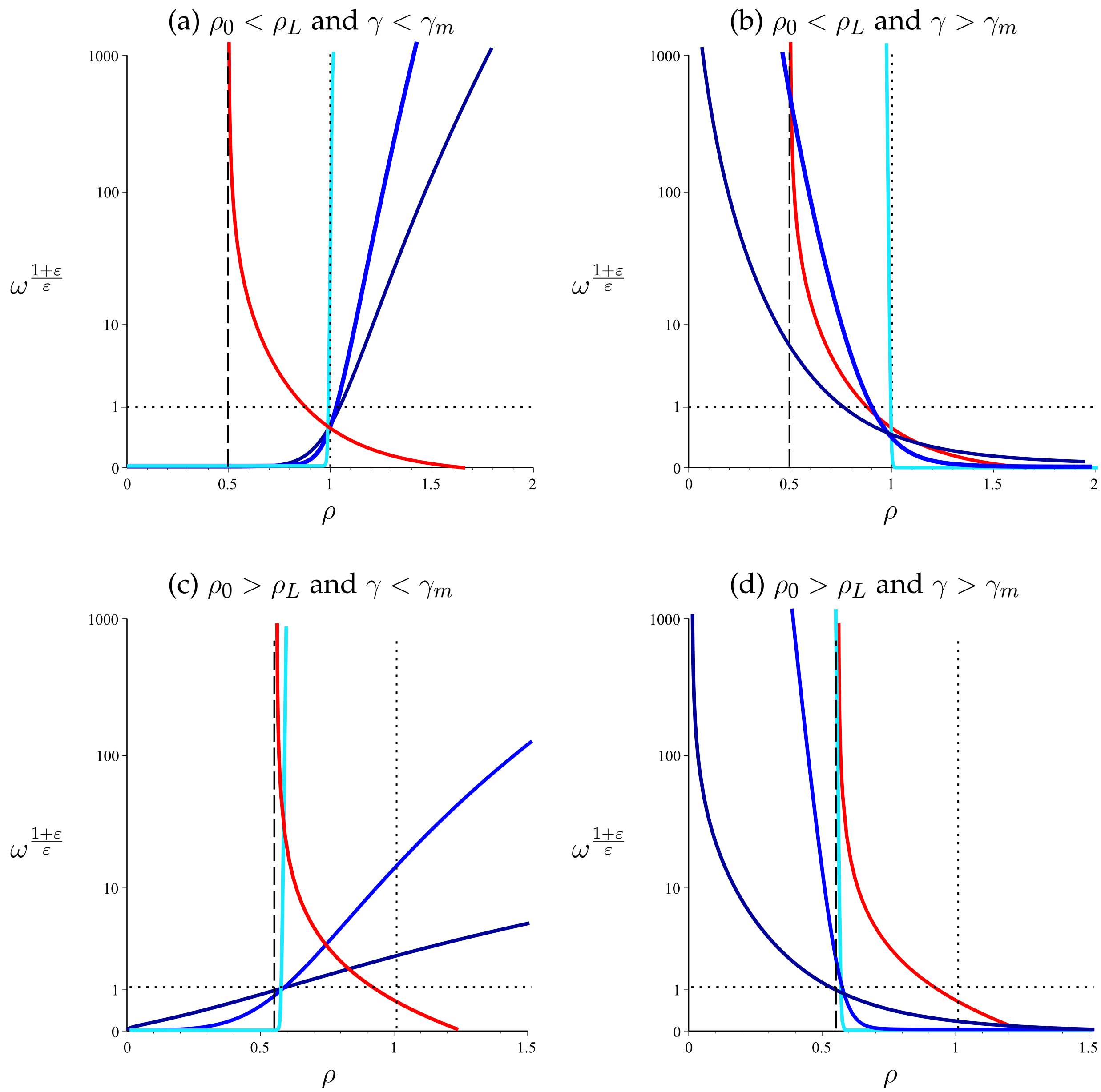
Proposition 1. *Assume $\gamma \neq \gamma_m \equiv \alpha/\varepsilon$. Then, a pair of relative land rents ρ^* and relative wages ω^* is an interior equilibrium if and only if it solves the two equations:*

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho) \equiv \frac{\phi\rho - 2\eta\phi\rho^{1+\frac{1}{\beta\varepsilon}} + (1+\phi^2)(1+\eta)}{(1+\eta)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - \eta\phi}, \tag{23}$$

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} = \begin{cases} g(\rho;\gamma) \equiv (c^\varepsilon\rho)^{\frac{\alpha\psi}{\alpha-\gamma\varepsilon}} \left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\right)^{\frac{\gamma\varepsilon}{\alpha-\gamma\varepsilon}\frac{1+\varepsilon}{\varepsilon}} & \text{when } \delta = 0, \\ \left(c\rho^{\frac{1-\alpha}{\beta\varepsilon}}\right)^{\frac{1+\varepsilon}{\alpha}} \left[\frac{\left(\omega\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\right)^\gamma + 2\delta}{\delta\left(\omega\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\right)^\gamma + (1+\delta^2)}\right] & \text{when } \delta > 0. \end{cases} \tag{24}$$

Proof: See in Appendix B.

Figure 1: Graphical demonstration of equilibrium for a range of parameter values.



Notes: *These figures illustrate equilibrium in twelve different cases. In all panels, the market-clearing locus, f is given by the red line. The blue lines describe the zero-profit locus, g. In the left two panels, darker colors of blue indicate smaller values of returns to scale, γ, and in the right two panels darker colors of blue indicate larger values of γ.*

Equations (23) and (24) are complicated, but the intuition behind them is simple. The expression for L in Lemma 1 follows immediately from utility maximization, while the expression for N requires land market clearing. Function f results from substituting L and N from Lemma 1 into the equilibrium condition (13). Thus, *f describes the locus of relative land rents and relative wages (ρ, ω) that satisfies cost minimization, utility maximization, and land market clearing*, but not zero profit. For simplicity, we call f the *market-clearing locus*. Because f does not require the zero-profit condition to hold, parameters that affect productivity directly, c, δ, and γ, do not appear in f.

The expression for g is more involved. It results from substituting equation (21) into (15). Equation (15) follows from cost minimization and the zero-profit condition, while equation (21) follows from utility maximization. Thus, *g describes the locus of relative land rents and relative wages (ρ, ω) satisfying cost minimization, utility maximization, and zero profits*, but not land market clearing. For simplicity, we call g the *zero-profit locus*.

Recalling the definition of a spatial equilibrium, a vector of relative land rents and relative wages (ρ,ω) that lies at the intersection of the market clearing and zero-profit loci is a spatial equilibrium. When $\delta = 0$, we can equate equations (23) and (24) to arrive at a single equation in ρ that determines the interior equilibria. In this case, we study the spatial equilibrium by studying the solution(s) of the equation,

$$f(\rho) = g(\rho; \gamma). \tag{25}$$

We show the existence of an interior equilibrium by showing that (25) has an interior solution. We determine the number of possible interior equilibria by determining the number of interior solutions of (25).

We cannot apply this solution method when $\delta > 0$ because no closed form expression for ω exists. The remainder of the section considers the case when $\delta = 0$. We postpone our treatment of the case when $\delta > 0$ to Section 6. Note that when $\gamma = \gamma_m$, the zero profit locus, g, is discontinuous, and so this case requires special attention.

Lemma 2 in Appendix C establishes that, as shown by the red line in all panels of Figure 1, the market clearing locus, $f(\rho)$, is a positive, continuous function that declines monotonically from a positive asymptote at ρ_m, to zero at ρ_M.

To develop intuition about the market-clearing locus, consider an increase in ω that reflects an increase in the central wage W_0. As W_0 increases, central firms substitute away from labor towards land, and central households spend more on residential land. For the central land market to clear, R_0 must increase and, therefore, ρ decreases. This gives the required negative relationship between ω and ρ along f. Mechanically, ω goes to zero or infinity as W_0 or W_1 approaches zero.

Lemma 2 also shows that there are two critical values of relative rent, ρ_m and ρ_M, such that the relationship between relative rent, ρ, and relative wage, ω, is negative along f when ρ varies between ρ_m and ρ_M. Values of ρ outside this interval imply negative wages, and so we focus on the interval $[\rho_m, \rho_M]$.

Let us now consider function g for $\delta = 0$. The left two panels of Figure 1, (a) and (c), describe g for three different values of $\gamma < \gamma_m$: dark blue the smallest γ, light blue the largest γ, medium blue in between. The right two panels, (b) and (d), are the same as the left, but for $\gamma > \gamma_m$. Here, the light blue line traces g for the smallest value of γ, dark blue uses the largest value, medium blue is intermediate value, and all three are greater than γ_m.

Lemma 3 in Appendix D establishes three properties of the zero-profit function, g. (i) When $\gamma < \gamma_m$, g is an increasing function that converges to an increasing step function as γ approaches γ_m from below. (ii) When $\gamma > \gamma_m$, g is a decreasing function that converges to a decreasing step function as γ approaches γ_m from above. (iii) The unique value of ρ at which the step occurs, denoted ρ_L, is strictly between zero and one.

Our intuition about the behavior of the zero-profit locus is based on the observation that, as γ increases, three quantities can adjust to preserve the zero-profit condition; wages, rents, and employment. For small values of γ, we can ignore changes in productivity, A_i. Indeed, when $0 < L_i < 1$, for γ small, L_i^γ is close to one unless L_i is close to zero. When wages go up in a location, preserving the zero-profit condition requires that the corresponding land rent must decline. This gives us a negative relationship between w and r, and thus, the positive relationship between ω and ρ that we see in the two left panels of Figure 1.

As γ increases beyond γ_m, TFP, A_i, becomes more sensitive to changes in γ because L_i^γ becomes more sensitive to small adjustments in employment L_i. Indeed, when $0 < L_i < 1$, L_i^γ is decreasing in γ. As a consequence, increases in γ

lead to larger changes of the same sign in A_i. Preserving the zero-profit condition now requires that wages and rents must move in the same direction. Thus, we have a positive relationship between w and r, or the negative relationship between ω and ρ that we see in the right two panels of Figure 1.

The singularity of function g arises when $\gamma = \gamma_m$, that is, when employment concentrates entirely in the center or periphery (on the zero-profit locus). In this case, the equilibrium condition (25) becomes invariant to changes in relative wages, which means that g is a step-function.

Figure 1 permits a graphical solution of equation (25), and hence a description of equilibrium for the particular examples illustrated in the figure. This figure suggests two main conclusions about equilibrium changes when returns to scale increase. First, when $\gamma < \gamma_m$, the market-clearing and zero-profit loci, f and g, cross exactly once for a positive value ρ, so a unique, interior equilibrium exists. Second, when $\gamma > \gamma_m$, f and g may cross more than once. Thus, $\gamma_m \equiv \alpha/\varepsilon$ is a threshold value of γ, below which there is a unique interior equilibrium, and above which multiple interior equilibria may occur.

Lemmas 2 and 3 guarantee that the location of the step in g lies to the left of the zero of f, so that $\rho_L < \rho_M$. However, the step in g can lie above the asymptote of f, as in the top two panels of Figure 1, or below, as in the bottom two panels. As a result, two cases may arise: ρ_L is larger or smaller than ρ_m. The top two panels of Figure 1, (a) and (b), describe the case where $\rho_m < \rho_L$, while the bottom two panels, (c) and (d), describe the opposite case.

It remains to determine what $\rho_m \gtrless \rho_L$ means. Lemma 4 in Appendix E establishes that $\rho_m < \rho_L$ if commuting costs are high *or* the demand for commercial land is sufficiently large relative to the demand for residential land. Conversely, if commuting costs are low *and* the demand for commercial land is low, then $\rho_m > \rho_L$.

Summing up, the critical value of γ (γ_m) and the three critical values of ρ (ρ_m, ρ_M and ρ_L) partition the parameter space into four regions with qualitatively different equilibrium behavior. These regions correspond to the panels of Figure 1 and are determined by whether $\gamma < \gamma_m$ or $\gamma_m < \gamma$ and by whether $\rho_m < \rho_L < \rho_M$ or $\rho_L < \rho_m < \rho_M$.

Stability

In the presence of multiple equilibria, it is common to appeal to stability as a selection device. One candidate, particularly relevant for the literature on quantitative spatial models, is to say that an equilibrium is stable if an iterative process converges to it. We show in Appendix F that this approach gives rise to unsuspected difficulties. In particular, the selected equilibrium may vary with the specification of the equilibrium conditions. Instead, we say that *a spatial equilibrium is stable if households want to return to the equilibrium when an arbitrarily small measure of them are displaced*. This definition of stability has three advantages. First, like our model, it is static and does not require an explicit description of time. Second, and unlike the other candidate definitions of stability, it has explicit behavioral foundations. Third, as we demonstrate, it is tractable.

Let ij and kl be two arbitrary location pairs; $ij = kl$ (location pairs are equal) when $i = k$ and $j = l$ hold simultaneously and distinct otherwise. We say that an equilibrium is *unstable* if, for some $ij \neq jk$, for any arbitrarily small $\Delta > 0$, there is a subset of individuals of mass Δ who strictly prefer the location pair kl, which differs from their utility-maximizing pair ij, when a perturbation moves them all to kl. In other words, the subset of individuals who have been moved away from ij do not want to move back. Otherwise, the equilibrium is *stable*.

The key issue is to determine the subset of individuals to use to check whether the equilibrium is unstable. In what follows, we assume that this subset is formed by individuals whose types are close to those of an individual indifferent between her equilibrium pair ij and another location pair kl.

Consider an equilibrium commuting pattern $\mathbf{s}^* \equiv (s^*_{ij})$, which could be interior or corner, and two location pairs, ij and kl, such that $ij \neq kl$ and $s^*_{ij} > 0$. We say that an individual $\nu \in [0,1]$ is *indifferent between ij and kl* if and only if

$$V^*_{ij}(\nu) = V^*_{kl}(\nu) \geq V^*_{od}(\nu), \tag{26}$$

for every location pair od such that $od \neq ij$ and $od \neq kl$. Lemma 5 in Appendix G establishes that such an individual always exists.

With this definition in place, we can now state our definition of stability formally.

Definition 1 *Consider an arbitrarily small subset of individuals of measure $\Delta > 0$ who choose ij and have types close to $\mathbf{z}(\nu) \in S_{ij}$ where ν is indifferent between ij and*

$kl \neq ij$. If this individual is strictly better off when she and her neighboring individuals are relocated from ij to kl, the spatial equilibrium is unstable. Otherwise, the spatial equilibrium is stable.

The motivation for this definition is as follows. If the relocation of a small group of almost indifferent individuals from ij to kl makes the indifferent agent strictly better off, then, by continuity there is a non-negligible subset of individuals who strictly prefer kl to ij. Hence, these individuals will never switch back to ij. On the contrary, if the indifferent individual never becomes strictly better off for any small subset, no other individual strictly prefers a different location pair. Hence, all the individuals will be willing to switch back to.

By relocating a small subset of individuals from ij to kl, the commuting pattern $\mathbf{s}$ becomes different from the equilibrium pattern $\mathbf{s}^*$. Hence, for our definition of stability to make sense, we must be able to compare the equilibrium and off-equilibrium utility levels. For this to be possible, we must determine the conditional equilibrium vectors of wages and land rents $\overline{\mathbf{W}}(\mathbf{s})$ and $\overline{\mathbf{R}}(\mathbf{s})$. We show in Appendix H that, for $\alpha > 1/2$, these vectors exist, are unique and continuous in $\mathbf{s}$.

4 Constant returns to scale

To begin, we consider a benchmark case when production is constant returns to scale, and there are no spillovers or first nature advantages. The following proposition characterizes the unique spatial equilibrium.

Proposition 2. *Assume constant returns to scale and no productivity spillovers ($\gamma = \delta = 0$). Then, there exists a unique equilibrium. This equilibrium has the following properties:*

(i) the equilibrium is interior;

(ii) wages are lower in the center than the periphery while rents are larger ($0 < \omega^* < 1, 0 < \rho^* < 1$);

(iii) if $(1-\alpha)/\alpha\beta > \varepsilon/(1+\varepsilon)$, then employment is larger in the center than the periphery ($\ell^* > 1$);

(iv) if $\varepsilon \to \infty$, then shares s^*_{ij} are equal to $1/3$ for $i = j$ and 0 for $i \neq j$.

Proof: See Appendix I.

Even without first nature advantages, spillovers, or returns to scale, agglomeration may occur. To understand why, consider the problem of a household faced with a choice of location and residence when wages and rents are the same in all locations. If we let $V = W/R^{\beta}$, then using (3), such a household's discrete choice problem is

$$\max_{ij} \left\{ \begin{array}{ccc} z_{-1,-1}V, & \frac{z_{-1,0}}{\tau}V, & \frac{z_{-1,1}}{\tau^2}V \\ \frac{z_{0,-1}}{\tau}V, & z_{0,0}V, & \frac{z_{0,1}}{\tau}V \\ \frac{z_{1,-1}}{\tau^2}V, & \frac{z_{1,0}}{\tau}V, & z_{1,1}V \end{array} \right\}.$$

This is the standard way of stating a discrete choice problem, except that we arrange the nine choices in a matrix so that the rows correspond to a choice of residence and columns to a choice workplace.

Suppose we restrict households to all choose a central residence. Because the distribution of idiosyncratic tastes is identical for all location pairs, the average payoff for a household at a central residence is

$$E\left(\max\left\{\frac{z_{0,-1}}{\tau}V, z_{0,0}V, \frac{z_{0,1}}{\tau}V\right\}\right) = \Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(1+\frac{2}{\tau^{\varepsilon}}\right)^{1/\varepsilon}V. \tag{27}$$

If, instead, we restrict households to choose a peripheral residence, then

$$E\left(\max\left\{z_{-1,-1}V, \frac{z_{-1,0}}{\tau}V, \frac{z_{-1,1}}{\tau^2}V\right\}\right) = \Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(1+\frac{1}{\tau^{\varepsilon}}+\frac{1}{\tau^{2\varepsilon}}\right)^{1/\varepsilon}V. \tag{28}$$

Because $\tau > 1$, it follows that the average payoff for a household in a peripheral residence is less than an average household in a central residence. By symmetry, exactly the same logic applies to the choice of employment. This occurs despite the fact that wages and rents are the same in all locations. In this sense, this discrete choice problem creates an *average preference* for residence and employment in the central location. Proposition 2 tells us that in equilibrium, these preferences are capitalized into lower central wages and higher central rents. While our model is simple, this phenomena appears to be general. If we exclude empirically uninteresting geographies like circles, most remaining location sets have a center in the sense of this example. We conclude that heterogeneous preferences together with commuting costs imply an average preference for residence and employment in the central location.

Against the two centralizing forces of average preferences are set two centrifugal forces. There is twice as much land in the periphery as the center. Because land contributes to utility and productivity, the scarcity of central land incentivizes the movement of both employment and residence to the periphery. Whether the center ends up relatively more specialized in residence or employment depends on which of the two activities has the highest demand for land, and this activity will locate disproportionately in the land-abundant periphery.

Proposition 2 makes this intuition precise. Agglomeration of production occurs at the center when

$$\frac{1}{\eta} = \frac{1-\alpha}{\alpha\beta} > \frac{\varepsilon}{1+\varepsilon}. \tag{29}$$

Recalling our discussion of η following equation (22), part (iii) of Proposition 2 says the center ends up relatively specialized in residence or employment depending on which of the two activities has the highest demand for land. With $\eta = 0.375$, the left hand side of equation (29) is about 2.67. It follows that equation (29) holds for all $\varepsilon > 0$, so that Proposition 2(iii) is probably the empirically relevant case.

It is tempting to think that the average preference for central work and residence is a response to commuting costs. This is not correct. From Proposition 2, when $\varepsilon \to \infty$ and preferences become homogeneous, we approach a city where each location is in autarky for any τ. Inspection of (27) and (28) shows why this occurs. When $\varepsilon \to \infty$, (27) and (28) are identical. As the same holds when $\tau = 1$, we may conclude that *both idiosyncratic preferences and commuting costs are necessary to create an average preference for residence and employment in the central location*.

5 First nature

We now turn attention to conventional sources of agglomeration. We begin with an examination of first nature productivity advantages. To isolate the role of first nature, we consider the case when production is constant returns to scale, $\gamma = 0$, and there are no spillovers, $\delta = 0$.

The following proposition characterizes the corresponding unique spatial equilibrium.

Proposition 3. *Assume constant returns and no productivity spillovers ($\gamma = \delta = 0$). Then, there exists a unique equilibrium and this equilibrium is interior for relative first* *nature advantage, c, positive and finite. The equilibrium employment distribution becomes more concentrated as the first nature productivity advantage $c = C_0/C_1$ increases. Furthermore, there exists a threshold level $\bar{c} > 0$ such that the equilibrium employment is*

(i) *equally distributed between the two peripheral locations, $(L_1, L_0, L_1) = (1/2, 0,1/2)$, when $c \to 0$;*

(ii) *higher in the periphery than the center, with $0 < L_0 < L_1$, when $0 < c < \bar{c}$;*

(iii) *uniformly distributed between center and periphery with $(L_1, L_0, L_1) = (1/3, 1/3, 1/3)$ when $c = \bar{c}$;*

(iv) *higher in the center than the periphery, $L_0 > L_1 > 0$, when $\bar{c} < c < \infty$;*

(v) *fully agglomerated at the center when $c \to \infty$.*

Proof: See Appendix J.

Proposition 3 shows that each value of c in $[0,\infty)$ uniquely determines a symmetric employment distribution. Furthermore, the employment ratio ℓ rises when first nature productivity is higher, while the two corner equilibria $\ell = 0$ and $\ell \to \infty$ arise when c takes its limit values 0 and ∞. This seems unsurprising, but requires two comments. First, Proposition 3 describes the relationship between first nature productivity and equilibrium employment patterns. It is silent about the residential pattern and commuting behavior. Second, Proposition 3 resembles Proposition 2 in Ahlfeldt *et al.* (2015). However, our result slightly extends the Ahlfeldt *et al.* result by mapping out the relationship between first nature advantages and equilibrium outcomes.

6 Spillovers

We now consider a city where, in addition to arbitrary first nature advantages, increasing returns or productivity spillovers operate at low levels. More specifically, we compare the impact of local increasing returns and technological spillovers in the vicinity of $\gamma = 0$ and $\delta = 0$ for arbitrary first nature productivity advantages. In doing so, we restrict attention to empirically relevant values of the two parameters.[4]

[4]Estimates of agglomeration economies suggest that IRS and spillover effects, γ and δ, are typically much less than one (Rosenthal and Strange, 2020).

Proposition 4. *Consider an economy with arbitrary relative first nature productivity, c, constant returns to scale, and no spillovers ($\gamma = \delta = 0$). Then,*

(i) there exists a unique threshold $\widehat{c} > 0$ such that increasing γ slightly above 0 increases central employment if $c > \widehat{c}$. When $c < \widehat{c}$, increasing γ slightly above 0 increases peripheral employment;

(ii) increasing δ from 0 to $\bar{\delta} \equiv (\sqrt{3} - 1)/2 \simeq 0.37$ raises central employment. However, increasing δ from $\bar{\delta}$ to 1 decreases central employment.

Proof: See Appendix K.

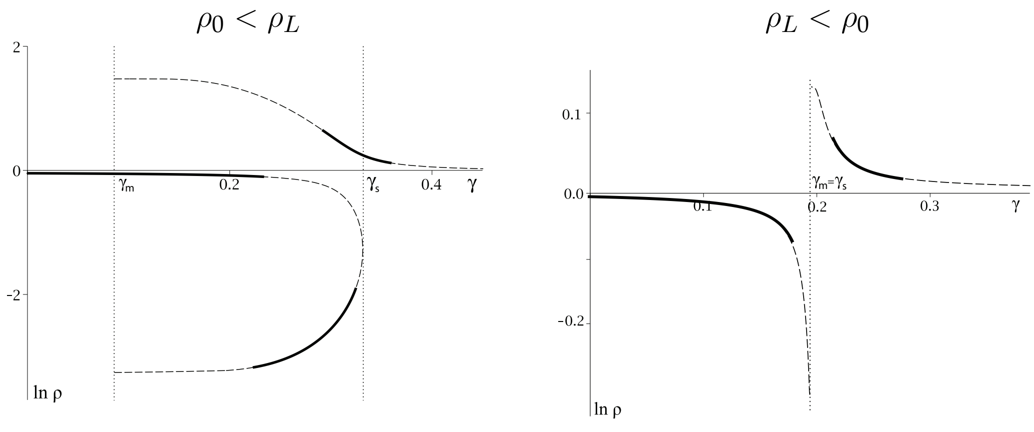
Increasing returns magnify the initial first-nature productive advantage of a location. Increasing γ raises the relative TFP $a(\gamma)$ if and only if the equilibrium under constant returns is such that $\ell > 1$. Thus, if first nature concentrates employment in the center, increasing returns increase this concentration. Furthermore, when $c > \widehat{c}$, the greater the relative initial advantage is, the greater is the impact of increasing returns.

There have been many efforts to measure spillovers empirically. However, these efforts do not permit a conclusion about whether or not the condition of Proposition 4(ii) is satisfied. In particular, Arzaghi and Henderson (2008) find that spillover effects in the advertising industry fall to almost zero over the space of a few city blocks. This suggests a value of δ smaller than 0.37. On the other hand, Carlino and Kerr (2015) estimate that, even for firms making software, the value of proximity to other firms falls to about 0.3 in $1 - 5$ miles, and falls more slowly for firms making fabricated metal. This does not seem to rule out values of δ above the 0.37 threshold of Proposition 4(ii). More generally, Proposition 4 suggests the importance of considering first nature when estimating the effects of spillovers. That "ship building" and "fish processing" are the two most dispersed industries of the 234 examined in Duranton and Overman (2005) probably has little to do with spillovers.

7 Increasing returns

We now turn attention to the role of increasing returns to scale. To focus attention on returns to scale, we rule out first nature advantages and spillovers,

Figure 2: Equilibrium correspondence between ρ and γ.



Notes: *In both panels the x-axis is returns to scale, γ, and the y-axis is the logarithm of relative land rent,* $\ln \rho$*. The left panel illustrates all interior equilibria as γ varies when $\rho_0 < \rho_L$. The right panel shows the case where $\rho_L < \rho_0$. Solid lines indicate stable equilibria and dashed lines indicate unstable equilibria, where stability is defined as in Section 3.*

and ask what happens as γ increases. In this case, we have $a(\ell) = \ell^\gamma$.

Proposition 5. *Assume that there are no relative first nature advantages and no spillovers ($\delta = 0$ and $c = 1$). If $\gamma > 0$, then*

(i) an interior equilibrium always exists.

(ii) there exist two corner equilibria where $(L_{-1},L_0,L_1) = (0,1,0)$ or $(1/2,0,1/2)$, but these equilibria are unstable;

(iii) in both corner and interior equilibria, each location hosts a positive mass of residents.

Proof: See Appendix L.

Part (i) of Proposition 5 is exactly what we would expect from inspection of Figure 1. Throughout the range of increasing returns, the market clearing locus, f, and the zero profit locus, g, always have at least one interior intersection.

The Fréchet distribution for the support of the z_{ij} in the indirect utility function (3) is unbounded, so we expect that every location will always have residents, as required by part (iii).

Regarding part (ii), we show in Appendix L that these corner equilibria are unstable. This result is easy to understand. Consider the agglomerated corner equilibrium $\mathbf{L}^* = (0,1,0)$. No single individual wants to move to, say, location 1 because her marginal productivity would be zero. This is why (0,1,0) is an equilibrium employment pattern. By contrast, when a small subset of households happens to be at $j = 1$, the marginal product of labor, and therefore the incomes, of individuals whose tastes are close to those of the indifferent individual are high. As a consequence, they do not want to move back to location 0. This means that we can ignore the corner equilibria.

Figure 2(a) depicts the equilibrium correspondence $\rho(\gamma)$. In Appendix M, Lemma 7 provides a simple test for checking the stability of any interior equilibria. Applying this test to the examples illustrated in Figure 2 allows us to determine the stability of each possible equilibrium for the relevant parameter values. We indicate stable equilibria with a heavy solid line, and unstable equilibria with a lighter dashed line.

With existence and stability established, we now turn to a characterization of equilibrium as γ increases from zero. We begin by introducing three domains of returns to scale by means of two thresholds γ_m and γ_s, which are defined below. We will show that these domains are associated with qualitatively different equilibrium behavior.

Definition 2 Increasing returns to scale are:

(i) weak if $0 < \gamma < \gamma_m \equiv \alpha/\varepsilon$;

(ii) moderate if $\gamma_m \leq \gamma \leq \gamma_s$;

(iii) strong if $\gamma > \gamma_s$.

As we describe above, $\gamma_m = \alpha/\varepsilon$ is the value of γ at which the zero-profit locus g switches from an increasing to a decreasing function of ρ. Because $\gamma_m = 0$ as $\varepsilon \to \infty$, it follows that the population must be heterogeneous for weak returns to scale to occur. Put differently, if households are homogeneous, the case of weak increasing returns is ruled out.

In a modern economy, the labor share of production, α, is about 0.6, while the range of commonly used estimates for ε is about [5,7]. Taking the ratio of these values, we have γ_m in [0.085,0.12].

Characterizing γ_s is more complicated. Taking the log of functions f and g and solving the equilibrium condition $\log f(\rho) = \log g(\rho; \gamma)$ for γ yields

$$\gamma(\rho) = \frac{\log f(\rho)}{\log \frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2} + \frac{\varepsilon}{1+\varepsilon} \log f(\rho)}. \tag{30}$$

The parameter γ_s is defined as the value of γ that maximizes this expression, subject to $\rho_m \leq \rho \leq 1$. Anticipating results below, $\gamma(\rho)$ is obtained by reversing the axes in Figure 2(a), and γ_s is defined as the maximum of $\gamma(\rho)$, the point where the two equilibrium branches with $\ln \rho < 0$ merge.[5]

To investigate the magnitude of γ_s, Figure 3 solves for the value of γ that maximizes equation (30) for a range of parameter values. Using plausible values for parameters, γ_s, evaluates to about 0.4.

Estimates of the wage elasticity of population for modern, developed country cities are often between 0.03 and 0.05, but other estimates suggest that larger values of γ are defensible. For the contemporary US, Glaeser and Gottlieb (2009) find that the elasticity of city level output to population is 0.13. Using data from six African countries in the early 21st century, Henderson and Turner (2020) find that the elasticity of household income to density is about 0.3, even after controlling for basic demographics. Examining about a century of rural and urban wage data for the US, Boustan *et al.* (2013) find that the urban wage premium is consistently around 0.3. Thus, values of γ between 0.05 and 0.30 find at least some empirical support (but large estimates probably partly reflect sorting and the rural to urban transition). This interval comfortably contains γ_m, and γ_s is near the top of this interval when other model parameters take defensible values. Thus, the domains of weak and moderate returns to scale seem empirically relevant while the domain of strong returns to scale cannot be ruled out with confidence.

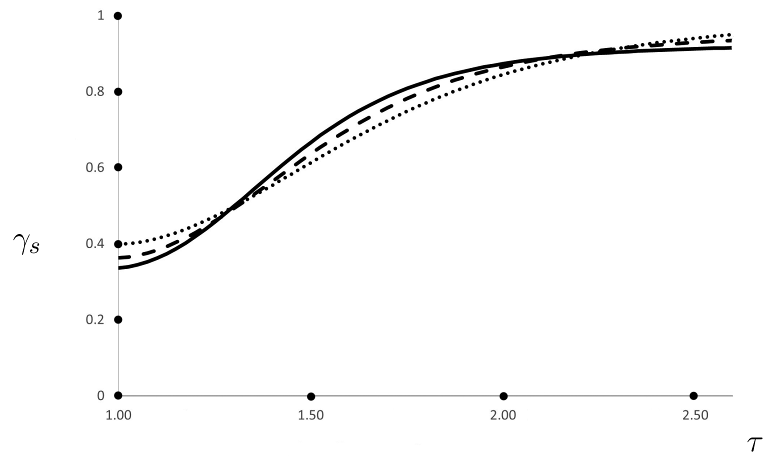
Weak increasing returns

The next proposition describes the equilibrium when increasing returns to scale are weak and ε is positive and finite.

Proposition 6. *Assume that there are no relative first nature advantages and no spillovers* ($c = 1$ *and* $\delta = 0$).

[5] Interestingly, we can show that $\gamma_m = \gamma_s$ when $\rho_m \geq \rho_L$ and $\gamma_m < \gamma_s$ when $\rho_m < \rho_L$. Therefore, the region of moderate increasing returns to scale does not exist unless the step in the zero profit locus is to the right of the asymptote of the market clearing locus, i.e., $\rho_m < \rho_L$.

Figure 3: Numerical evaluation of γ_s



Notes: *This figure plots the γ_s, the value of γ that maximizes equation* (30). *The figures are drawn for $\alpha = 0.6$ and $\beta = 0.15, \delta = 0, c = 1, \tau = 1.15$ and $\varepsilon = 5$ (dotted), 6 (dashed) and 7 (solid). For iceberg commuting cost, τ in the empirically relevant range of $(1,1.5)$, γ_s is around 0.4.*

(i) If $0 < \gamma < \gamma_m$, then there is a unique interior equilibrium;

(ii) If $(1-\alpha)/\alpha\beta > \varepsilon/(1+\varepsilon)$ holds, then equilibrium employment in the center is greater than in the periphery. Furthermore, central employment, rents and wages are all increasing in γ. That is,

$$\frac{d\ell^*}{d\gamma} > 0, \quad \frac{d\rho^*}{d\gamma} < 0, \quad 0 < \frac{d\omega^*}{d\gamma}.$$

Proof: See Appendix N.

In the region of weak increasing returns, equilibrium is largely determined by the same forces that operate when the technology displays constant returns. That is, the average preferences for central work and residence draw activity into the center and the relative abundance of peripheral land pulls it out. As scale economies increase, the central location becomes increasingly attractive for employment, the central land price capitalizes higher central productivity, and the central wage rises in response to the increase in the marginal product of labor.

The comparative statics in Proposition 6 holds whenever $0 < \gamma < \gamma_m$. The generality of this result conceals the fact that distinct equilibrium regimes arise when $\rho_m < \rho_L$ and $\rho_L < \rho_m$. When $\rho_m < \rho_L$, high commuting costs encourage households to work where they live or land hungry production faces pressure to disperse to the periphery (or both). An equilibrium in such an economy has low levels of commuting and dispersed production. Thus, when $\rho_m < \rho_L$, low levels of increasing returns do not lead to equilibrium cities where employment or residence is highly concentrated in either the center or periphery. Panel (a) of Figure 1 illustrates this case. Formally, as γ approaches γ_m, the zero-profit locus converges to an increasing step function with the step at ρ_L. Hence, the curves f and g must cross near ρ_L as γ increases towards γ_m. Because $\rho_m < \rho_L$, it follows that this intersection must occur when f is away from its asymptote at ρ_m. Therefore, the equilibrium value of ω grows with γ but remains bounded.

In contrast, when $\rho_L \leq \rho_m$, low commuting costs allow households to separate work and residence locations in response to a small wage premium, and productivity is not sensitive to the relatively abundant land of the periphery. In this case, increasing returns compounds the average preference for central employment to concentrate employment in the center, and households are able to cheaply disperse their residences to the land-abundant periphery. An equilibrium in such an economy involves concentrated employment and high levels of commuting. Thus, when $\rho_L < \rho_m$, low levels of increasing returns lead to equilibrium cities where employment is highly concentrated in the center and residence in the periphery. The monocentric city arises endogenously. Formally, when $\rho_L \leq \rho_m$, as γ approaches γ_m, the intersection of f and g occurs near the asymptote of f. As a result, the value of ω at which the two curves intersect becomes large.

Figure 2 shows the equilibrium relationship between γ and $\ln \rho$ for numerical examples satisfying $\rho_m < \rho_L$ in panel (a) and $\rho_L < \rho_m$ in panel (b). In both panels, the x-axis is γ and the y-axis is $\ln \rho$. Both figures show all interior equilibria, but not the corner equilibria. Both figures indicate stable equilibria with a solid line and unstable equilibria with a dashed line. Consistent with our results in Propositions 2 and 6, Figure 2 shows that ρ decreases from near one as γ increases in a neighborhood of zero. As γ increases towards γ_m, we see that ρ continues to decrease. Comparing panels (a) and (b) we see the two distinct equilibrium regimes that arise as γ increases toward the threshold of the weak increasing returns domain when $\rho_m < \rho_L$ and $\rho_L < \rho_m$.

By Proposition 5, interior equilibria always exist. Lemma 7 in Appendix M provides a simple test for checking the stability of any interior equilibria. Applying this test to the examples illustrated in Figure 2 allows us to determine the stability of each possible equilibrium for the relevant parameter values. In Figure 2, we indicate stable equilibria with a heavy solid line, and unstable equilibria with a lighter dashed line. While our results do not permit general conclusions about the stability of equilibria, they demonstrate that stable equilibria need not exist and that multiple stable equilibria are possible.

Moderate increasing returns

When $\gamma > \gamma_m$, the market clearing locus, f, remains unchanged, but the zero-profit locus, g, changes from an increasing to a decreasing function. When f and g are both decreasing, they need not cross at all, and may cross more than once. Thus, zero or many equilibria are possible. Proposition 5 establishes that an equilibrium exists. Figure 1 suggests that for intermediate values of γ, that f and g cross one or three times. The following proposition formalizes this intuition.

Proposition 7. *Assume that there are no relative first nature advantages and no spillovers* ($\delta = 0$ *and* $c = 1$). *If* γ *is slightly above* γ_m *and*

(i) *if commuting costs are high or the demand for commercial land is large* ($\rho_m < \rho_L$), *then there are three equilibria:* (1) *central wages and rents are larger than in the periphery* ($\omega_1^* > 1$ *and* $\rho_1^* < 1$); (2) *close to the equilibrium that occurs under weak increasing returns with central wages and rents larger or smaller than in the periphery* ($\omega_2^* \lesseqgtr 1$ *and* $\rho_2^* \lesseqgtr 1$); (3) *central wages and rents are smaller than in the periphery* ($\omega_3^* < 1$ *and* $\rho_3^* > 1$). *Furthermore, as returns to scale decrease towards the weak/moderate threshold* ($\gamma \to \gamma_m$), *employment in* (1) *concentrates in the periphery; employment in* (2) *remains interior; and employment in* (3) *concentrates in the center;*

(ii) *if commuting costs are low and the demand for commercial land is small* ($\rho_m > \rho_L$), *then there is a unique interior equilibrium* (ρ^*, ω^*) *such that central wages and rents are smaller than in the periphery* ($\omega^* < 1$ *and* $\rho^* > 1$). *Furthermore, as returns to scale decrease towards the weak/moderate threshold* ($\gamma \to \gamma_m$), *equilibrium employment concentrates in the periphery.*

Proof: See Appendix O.

The logic underlying part (i) of Proposition 7 is visible in panel (b) of Figure 1. The medium blue line describes the case of moderate increasing returns. In this case, g crosses f three times. At the first intersection point, we have $\rho_1^* < 1$ and

$\omega_1^* > 1$; at the second, we see that ρ_2^* approaches ρ_L as γ decreases toward γ_m; at the third intersection point, we have $\rho_3^* > 1$ and $\omega_3^* < 1$. The value ω_1^* (resp., ω_3^*) in turn requires that employment occurs primarily in the center (resp., periphery).

The light blue line in panel (b) describes g when γ is just above γ_m. As γ approaches this threshold, for one of the two new equilibria, ω grows without bound (and occurs outside the frame of the figure), while ω approaches zero in the other equilibrium. That is, just above the threshold, these two equilibria approach corner patterns where all employment is either central or peripheral.

The corresponding logic for part (ii) of Proposition 7 is visible in panel (d) of Figure 1. As in panel (b), the medium blue line describes the case of moderate increasing returns. In this case, g crosses f only once. The light blue line in panel (b) describes g when γ is just above γ_m. As γ approaches this threshold, the single intersection of f and g occurs at progressively larger values of ω as g converges to a decreasing step function. In the limiting case, as γ approaches γ_m from above, the single equilibrium occurs when all employment is concentrated in the periphery. Figure 2 summarizes the results of Proposition 7.

That part (i) of Proposition 7 establishes the emergence of multiple equilibria seems unsurprising. We expect sufficiently strong increasing returns to give rise to multiple equilibria. Part (i) also describes an equilibrium branch which continues behavior from the weak increasing returns case. This is the central, interior equilibrium branch in panel (a) of Figure 2. That the logic governing behavior in the case of weak increasing returns should sometimes survive a small increase in γ above γ_m also seems unsurprising.

We see in part (ii) that multiple equilibria need not emerge as returns to scale increase. When commuting costs are low and land is less productive, i.e. $\rho_m > \rho_L$, there is only a single equilibrium when γ is weak or moderate. Thus, increasing returns is necessary for multiple equilibria, but not sufficient.

Propositions 6 and 7 together imply a discontinuous change in the only possible equilibrium city when $\rho_L < \rho_m$ and γ varies around γ_m. This discontinuity is clearly visible in panel (b) of Figure 2. From Figure 1, it is clear that this discontinuity follows from the fact that the zero-profit locus switches from an increasing to a decreasing step function around the threshold γ_m.

These results require three comments. First, notice that Proposition 7 characterizes equilibrium just above the threshold separating weak and moderate increasing returns, γ_m. It is natural to expect that the behavior we observe near γ_m persists throughout the full range of moderate increasing returns, as in the example illustrated in Figure 2. In fact, we cannot rule out the possibility of more complicated equilibrium behavior for values of γ just below γ_s, although we have not found a counter example to contradict the conjecture that the results of Proposition 7 hold throughout the range of moderate increasing returns.

Second, Proposition 7 establishes qualitatively different equilibrium behavior around $\gamma = \gamma_m$ when $\rho_m < \rho_L$ and conversely. Recalling that $\rho_L < \rho_m$ describes the case when commuting costs and the land share of production are both low, this is the case when we expect the location of employment to be more sensitive to changes in returns to scale. This intuition is consistent with our finding in Proposition 7.

Third, when commuting costs are high or land is more productive, i.e. $\rho_m < \rho_L$, part (i) of Proposition 7 establishes the existence of three equilibria when γ is not too far above the γ_m threshold. As γ decreases towards γ_m employment in these equilibria converges to the corners $(L_{-1},L_0,L_1) = (1/2,0.1/2)$ or $(0,1,0)$. Reversing these statements, in these equilibria, as returns to scale increase, employment becomes more uniformly distributed across locations. For these equilibria, increases in returns to scale does not lead to agglomeration. It leads to its opposite.

Strong increasing returns

Careful inspection of Figure 1 shows that the market clearing and zero profit curves cross only once and that a single interior equilibrium persists when γ is larger than a threshold γ_s, regardless of whether $\rho_m < \rho_L$ or $\rho_L < \rho_m$. The following theorem extends and makes precise this intuition.

Proposition 8. *Assume that there are no relative first nature advantages and no spillovers ($\delta = 0$ and $c = 1$). If $\gamma > \gamma_s$, then there exists a unique interior equilibrium. Furthermore, as returns to scale γ increase and tend to ∞, ℓ^* decreases and the distribution of employment converges toward the uniform distribution.*
Proof: See Appendix P.

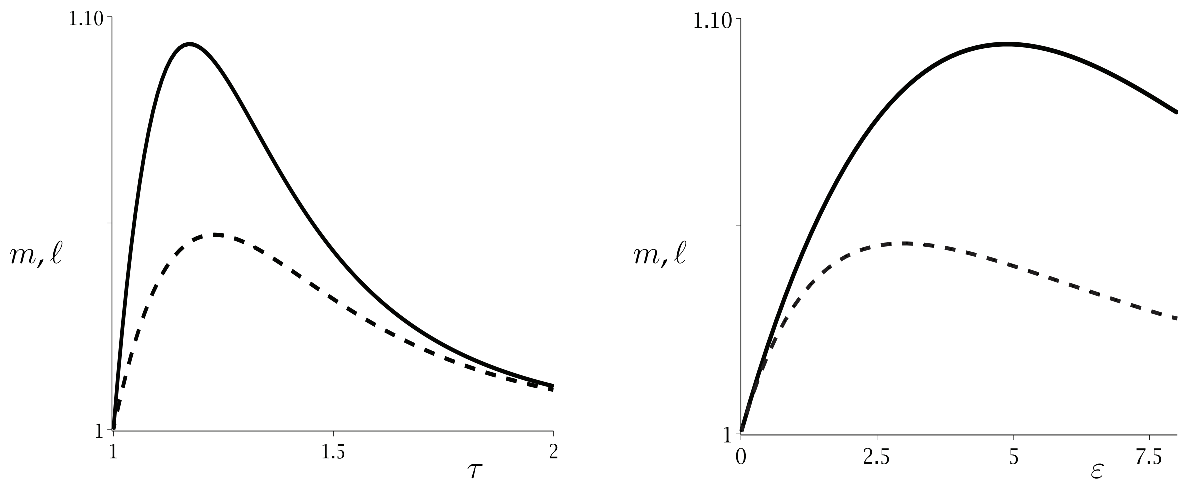
Equilibrium requires that profits be zero everywhere, that households choose their most preferred workplace-residence pair, that firms choose inputs to minimize costs, and that land markets clear. An increase in returns to scale immediately upsets the zero-profit condition as productivity increases more rapidly in more densely populated places, changing A_0 and A_1 in equation (14). Cost minimization dictates the corresponding changes in wages. With wages determined, two degrees of freedom remain to restore the zero-profit condition, a change in rent and, recalling that $A_i = L_i^{\gamma}$, a change in population, L_i. In particular, an increase in rent or a decrease in population can both reduce profits after an increase in the strength of agglomeration economies. Proposition 8 shows that equilibrium increases in wages and rents are not sufficient to preserve the zero-profit condition when increasing returns are sufficiently high. Restoring the zero-profit condition is accomplished by dispersing employment to reduce wage and rent at the center. In other words, *strong increasing returns act as a dispersion force*.

To sum up, Proposition 6 shows that for low levels of returns to scale, increases in γ lead to *increased* concentration of employment in the center. We see in Proposition 7(i) that increases in γ lead to *decreases* in employment concentration along the equilibrium branches where employment is highly concentrated in the center or the periphery. Proposition 8 shows that beyond γ_s, increases in γ lead to *decreases* in employment concentration for the only equilibrium that persists at high levels of returns to scale. Thus, increasing returns is not an agglomeration force over its entire possible range; for $\gamma > \gamma_m$, increases γ may cause employment to disperse. For $\gamma > \gamma_s$, increases to γ must cause employment to disperse.

The left panel of figure 2 illustrates this behavior. In the region of moderate returns to scale, both of the extreme equilibrium branches exhibit increasing dispersion as returns to scale increases. Above the threshold of strong returns to scale, the single surviving equilibrium branch also exhibit increased dispersion as γ increases. Importantly, these equilibria are sometimes stable. This establishes that the behavior described by Proposition 7(ii) and Proposition 8 cannot be disregarded because the relevant equilibria are unstable.

Inspection of Figure 1 makes it clear why this result occurs. As γ increases beyond γ_m, the zero-profit locus, g, diverges from the step function $g(\rho,\gamma_m)$ and as this occurs, the intersection of f and g occurs at values of ρ that are progressively nearer to one.

Figure 4: Employment and residence ratios as commuting cost and preference heterogeneity vary.



Notes: *In both panels, the heavy black line is equilibrium employment centrality, $\ell = L_0/L_1$, the dashed line is residence centrality, $m = M_0/M_1$ and parameters values are $\gamma = 0.07$, $\alpha = 0.6, \beta = 0.15, \delta = 0, c = 1$. (Left) $\varepsilon = 6$ and τ varies along the x-axis. (Right) $\tau = 1.15$ and ε varies along the x-axis. In all cases, parameters lie in the region of weak increasing returns ($\gamma < \gamma_m$). For an empirically relevant part of the parameter space, the figures show a non-linear relationship between the centralization of employment and residence and commute costs (Right) and preference heterogeneity (Left).*

8 Commuting cost and preference dispersion

We now consider how interior equilibria vary as commuting cost τ or preference dispersion ε change . Because τ and ε often appear together in the analysis as $\phi = \tau^{-\varepsilon}$, we may expect the comparative statics for the two parameters to be similar.

Figure 4a illustrates the equilibrium employment ratio ℓ and residence ratio, m, as τ varies in the case of weak increasing returns: the concentration of employment and residence in the center is increasing in commuting costs for low levels of τ and decreasing for high levels of τ. Equilibrium employment and residence becomes uniform as τ approaches either one or infinity. The peak of the employment ratio locus occurs around $\tau = 1.2$, where commuting results in a 20% utility penalty. For reference, Redding and Turner (2015) report an average round trip commute of about 50 minutes for an average American, this is about 12% of an eight hour work day. This calculation suggests that the complicated comparative statics illustrated in Figure 3 could well be empirically relevant.

It is easy to see that $\phi = 1$ when $\tau = 1$, while $\phi = 0$ when $\tau \to \infty$. In these two cases, equations (27) and (28) show that the average payoff of the choice of central versus peripheral workplace converge toward each other. To show this, setting $\phi = 0$ and $\phi = 1$ in equations (23) and (24), we obtain:

$$\begin{aligned} f(\rho)|_{\phi=0} &= \rho^{-1-\frac{1}{\beta\varepsilon}}, \qquad f(\rho)|_{\phi=1} = \frac{\rho - 2a\rho^{1+\frac{1}{\beta\varepsilon}} + 2(1+a)}{(1+a)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\rho^{\frac{1}{\beta\varepsilon}} - a}, \\ g(\rho;\gamma)|_{\phi=0} &= \rho^{\frac{1}{a}+\frac{\gamma}{\alpha\varepsilon}}\rho^{\frac{\gamma\varepsilon/\alpha}{1-\gamma\varepsilon/\alpha}\frac{1+\varepsilon}{\varepsilon}}, \qquad g(\rho;\gamma)|_{\phi=1} = \rho^{\frac{b}{1-\gamma\varepsilon/\alpha}}. \end{aligned}$$

Evaluating these functions at $\rho = 1$ shows that $f(1)|_{\phi=0} = g(1;\gamma)|_{\phi=0} = 1$ and $f(1)|_{\phi=1} = g(1;\gamma)|_{\phi=1} = 1$. Hence, $\rho^* = 1$ is an interior equilibrium when $\phi = 1$ and when $\phi = 0$.

Figure 4b illustrates the equilibrium employment ratio ℓ and residence ratio, m, as ε varies, also in the case of weak increasing returns. We see that agglomeration occurs only when ε takes intermediate values. Letting $\varepsilon \to \infty$ in $f(\rho)|_{\phi=0} = g(\rho;\gamma)|_{\phi=0}$, we obtain $\rho^{-1} = \rho^{\frac{1}{a}-1}$ whose unique interior solution is $\rho^* = 1$. It then follows from equation (23) that $\omega^* = 1$, which implies $\ell^*(1) = m^*(1) = h^*(1) = n^*(1) = 1$. Using equations (17) and (18), it is easy to show that $s^*_{ij} = 0$ for $i \neq j$ and $s^*_{ii} = 1/3$. Thus, when households no longer care about where they live and work, they focus on their own consumption only.

By contrast, as $\varepsilon = 0$, returns to scale must be weak because the threshold $\gamma_m \to \infty$. The uniform pattern also emerges as the population becomes infinitely heterogeneous. As taste heterogeneity over pairwise choices becomes increasingly important relative to commuting costs and, in the limit, households ignore land price and wage differences, and the distribution of households across pairs is uniform. Since the distribution of types is the same across locations, households must be uniformly distributed across locations.

Thus, as ε goes to zero or infinity, the distribution of residence and employment becomes uniform, but the city functions differently. When $\varepsilon = 0$, we have an extreme form of urban sprawl where many people commute and there is no city center. Formally, when $\varepsilon = 0$, the payoffs for each of the nine location pairs become identical ($s^*_{ij} = 1/9$). By contrast, when $\varepsilon \to \infty$ we have a city of backyard capitalists in which nobody commutes. Indeed, the payoff (28) attached to off-diagonal pairs goes to zero ($s^*_{ij} = 0$ for $i \neq j$).

Summing up, these results, and those discussed in Section 3, show that introducing heterogeneity in households' preferences for workplace-residence pairs is not an innocuous assumption. On the contrary, it has important effects on the properties of the spatial equilibrium.

Three remarks are in order. First, in the monocentric city model, decreases in commuting costs lead households to spread out. In our heterogenous household model, comparative statics on commuting costs are not monotone. Second, quantitative exercises often evaluate the effects of counterfactual changes in commuting costs. To the extent that these counterfactual exercises are comparative statics of commuting costs, our results suggest that the qualitative features of such counterfactual exercises may change sign in response to changes in parameters according to the value of ε. Finally, we note that our comparative statics on preference heterogeneity are similar to those from new economic geography. Combining results from Krugman(1991) and Tabuchi and Thisse(2002), economic activity is dispersed in economies with homogenous households, concentrated for intermediate levels of heterogeneity, and concentrated again for economies with very heterogenous households. In the NEG framework, as in our model, the addition of idiosyncratic preferences for locations leads qualitatively different equilibria.

9 Conclusion

Understanding how people arrange themselves when they are free to choose work and residence locations, when commuting is costly, and when some economic mechanism rewards the agglomeration of employment, is one of the defining problems of urban economics. We address this problem by combining a discrete choice model of location, the stylized geographies of classical urban economics, and a production function that allows for first nature advantages, increasing returns, and productivity spillovers. We provide a complete description of equilibria in much the parameter space.

Besides accounting identities, an equilibrium must satisfy two main conditions: all households choose their most preferred workplace-residence pair and profits must be zero everywhere. Of these two, the first is familiar from widely used discrete choice models of spatial equilibrium. The second is less well studied and has two surprising implications. First, equilibrium agglomeration of employment is first increasing and then decreasing in the strength of returns to scale. When increasing returns to scale are strong enough, the zero profit condition is preserved, in part, by dispersing employment. Second, productivity spillovers can act to disperse employment. Productivity spillovers allow firms to benefit from high productivity locations without paying the rent and wage premium required to locate in them. It is enough to be near. Of the three conventional foundations for economic agglomerations that we consider, only the comparative statics for first nature behave as expected: employment concentrates where first nature productivity is greater.

Despite its wide use, our conventional description of preference heterogeneity implies a previously unnoticed foundation for agglomeration. A population of households with heterogenous preferences over workplace-residence pairs has an average preference for central work and residence. Absent any property of production that rewards the concentration of employment, a city comprised of such households has denser central employment and residence.

Even in a simple setting like ours, the relationship between economic fundamentals and equilibrium is complicated. This is true throughout the parameter space, but particularly in the region of moderate returns to scale. As we see in Figure 2, it is in this region where multiple interior equilibria arise, where equilibrium comparative statics can be discontinuous, and where increasing to returns to scale begins to disperse employment. Our back of the envelope calculations show that when other parameters of the model take defensible values, the threshold values for return to scale, γ_m and γ_s, are respectively central and at the boundary of the set of plausible values for γ. This suggests that none of the equilibrium behavior that we describe can be ruled out a priori in applications of QSM.

Our results also have implications for research based on quantitative spatial models. Two observations will illustrate. First, Heblich *et al.* (2020) estimate a model with many features in common with ours. They demonstrate that a reduction in commuting costs following the opening of the subway in 19th century London precipitates a dramatic reorganization of the city. Loosely, prior to the subway, employment and residence were dispersed and commuting was relatively rare, and after the subway, the locations of residence and employment separated as people commuted to central employment from peripheral residences. According to the discussion following Proposition 6, high commuting costs encourage households to work where they live, while low commuting costs allow households to separate work and residence locations. Thus, the phenomena observed in London correspond closely with comparative static in our model in a large part of the parameter space. This argues for the external validity of the comparative statics evaluated in Heblich *et al.* (2020). At a minimum, their finding does not require all of the many peculiarities of London to arise.

Second, as a rule, quantitative spatial models often share many features with the one considered here, and so may be expected to exhibit at least some of the same complicated behavior. The possibility of complex behavior in a neighborhood of the boundary between the weak and moderate returns to scale together with the empirical relevance of this threshold suggests that efforts to investigate the possibility of multiple equilibria are appropriate. We have shown, that convergence of fixed point algorithms fails to serve as an equilibrium selection criterion under multiple equilibria because it is not robust to the algebraic form of the equilibrium conditions. Thus, an investigation of multiple equilibria appears to require new techniques.

References

Ahlfeldt, G. M., Redding, S. J., Sturm, D. M., and Wolf, N. (2015). The economics of density: Evidence from the berlin wall. *Econometrica*, 83(6):2127–2189.

Allen, T. and Arkolakis, C. (2014). Trade and the topography of the spatial economy. *The Quarterly Journal of Economics*, 129(3):1085–1140.

Anas, A. (1983). Discrete choice theory, information theory and the multinomial logit and gravity models. *Transportation Research Part B: Methodological*, 17(1):13–23.

Anderson, S. P., de Palma, A., and Thisse, J.-F. (1992). *Discrete choice theory of product differentiation*. MIT Press.

Arzaghi, M. and Henderson, J. V. (2008). Networking off madison avenue. *The Review of Economic Studies*, 75(4):1011–1038.

Ben-Akiva, M. E. and Lerman, S. R. (1985). *Discrete choice analysis: theory and application to travel demand*, volume 9. MIT press.

Berliant, M., Peng, S.-K., and Wang, P. (2002). Production externalities and urban configuration. *Journal of Economic Theory*, 104(2):275–303.

Boustan, L. P., Bunten, D. M., and Hearey, O. (2013). Urbanization in the united states, 1800-2000. Technical report, National Bureau of Economic Research.

Carlino, G. and Kerr, W. R. (2015). Agglomeration and innovation. *Handbook of regional and urban economics*, 5:349–404.

Ciccone, A. and Hall, R. E. (1996). Productivity and the density of economic activity. *The American Economic Review*, 86(1):54–70.

de Palma, A., Ginsburgh, V., Papageorgiou, Y. Y., and Thisse, J.-F. (1985). The principle of minimum differentiation holds under sufficient heterogeneity. *Econometrica*, pages 767–781.

Dong, X. and Ross, S. L. (2015). Accuracy and efficiency in simulating equilibrium land-use patterns for self-organizing cities. *Journal of Economic Geography*, 15(4):707–722.

Duranton, G. and Overman, H. G. (2005). Testing for localization using micro-geographic data. *The Review of Economic Studies*, 72(4):1077–1106.

Duranton, G. and Puga, D. (2004). Micro-foundations of urban agglomeration economies. In *Handbook of regional and urban economics*, volume 4, pages 2063–2117. Elsevier.

Eaton, J. and Kortum, S. (2002). Technology, geography, and trade. *Econometrica*, 70(5):1741–1779.

Fujita, M. (1989). *Urban economic theory: Land use and city size*. Cambridge University Press.

Fujita, M. and Ogawa, H. (1982). Multiple equilibria and structural transition of non-monocentric urban configurations. *Regional Science and Urban Economics*, 12(2):161–196.

Glaeser, E. L. and Gottlieb, J. D. (2009). The wealth of cities: Agglomeration economies and spatial equilibrium in the united states. *Journal of economic literature*, 47(4):983–1028.

Heblich, S., Redding, S. J., and Sturm, D. M. (2020). The making of the modern metropolis: evidence from london. *The Quarterly Journal of Economics*, 135(4):2059–2133.

Henderson, J. V. and Turner, M. A. (2020). Urbanization in the developing world: too early or too slow? *Journal of Economic Perspectives*, 34(3):150–173.

Krugman, P. (1991). Increasing returns and economic geography. *Journal of political economy*, 99(3):483–499.

Lucas, R. E. and Rossi-Hansberg, E. (2002). On the internal structure of cities. *Econometrica*, 70(4):1445–1476.

Matsuyama, K. (1991). Increasing returns, industrialization, and indeterminacy of equilibrium. *The Quarterly Journal of Economics*, 106(2):617–650.

Ogawa, H. and Fujita, M. (1980). Equilibrium land use patterns in a nonmonocentric city. *Journal of Regional Science*, 20(4):455–475.

Redding, S. J. and Rossi-Hansberg, E. (2017). Quantitative spatial economics. *Annual Review of Economics*, 9:21–58.

Redding, S. J. and Turner, M. A. (2015). Transportation costs and the spatial organization of economic activity. *Handbook of regional and urban economics*, 5:1339–1398.

Rosenthal, S. S. and Strange, W. C. (2020). How close is close? the spatial reach of agglomeration economies. *Journal of economic perspectives*, 34(3):27–49.

Tabuchi, T. and Thisse, J.-F. (2002). Taste heterogeneity, labor mobility and economic geography. *Journal of Development Economics*, 69(1):155–177.

Appendix

A. Proof of Lemma 1

As commuting flows (7) are uniquely determined by ρ and ω, it suffices to express the equilibrium patterns — the labor pattern (L_0,L_1), the residential population pattern (M_0,M_1), the housing pattern (H_0,H_1), the commercial land-use pattern (N_0,N_1), the wage pattern (W_0,W_1), and the land rent pattern (R_0,R_1) — as functions of ρ, ω, and (s_{ij}), or equivalently, as functions of r, w and (s_{ij}).

Labor and population patterns: Using (7) and (8) yields (17) and (18).

Land-use patterns: Consider the complementary slackness conditions for producer's profit maximization:

$$\left(\alpha A_i L_i^{\alpha-1} N_i^{1-\alpha} - W_i\right) L_i = 0, \quad \text{with } \alpha A_i L_i^{\alpha-1} N_i^{1-\alpha} - W_i \leq 0 \text{ and } L_i \geq 0,$$

$$\left[(1-\alpha) A_i L_i^{\alpha} N_i^{-\alpha} - R_i\right] N_i = 0, \quad \text{with } (1-\alpha) A_i L_i^{\alpha} N_i^{-\alpha} - R_i \leq 0 \text{ and } N_i \geq 0.$$

These complementary slackness conditions imply that

$$A_i L_i^{\alpha} N_i^{1-\alpha} = \frac{W_i L_i}{\alpha} = \frac{R_i N_i}{1-\alpha}.$$

Hence, the demands for commercial land are given by

$$N_0 = \frac{1-\alpha}{\alpha}\frac{W_0 L_0}{R_0}, \qquad N_1 = \frac{1-\alpha}{\alpha}\frac{W_1 L_1}{R_1}.$$

Using (8), we obtain expressions for the commercial land demands:

$$N_0 = \frac{1-\alpha}{\alpha}\frac{W_0}{R_0}(s_{00} + 2s_{10}), \qquad N_1 = \frac{1-\alpha}{\alpha}\frac{W_1}{R_1}\left[(1+\phi^2)s_{11} + s_{01}\right]. \tag{A.1}$$

Next, plugging the commuting flows (7) into the market demand functions for residential land, $H_i \equiv \sum_j s_{ij}\frac{\beta W_j}{R_i}$, and using $w = W_0/W_1$, we obtain expressions for the residential land demands:

$$H_0 = \beta\frac{W_0}{R_0}\left(s_{00} + 2w^{-1}s_{01}\right), \qquad H_1 = \beta\frac{W_1}{R_1}\left[(1+\phi^2)s_{11} + ws_{10}\right]. \tag{A.2}$$

Computing the ratios, H_i/N_i, for $i = 0,1$, and using (10),

$$\frac{H_0}{N_0} = \frac{1-N_0}{N_0} = \eta\frac{s_{00} + 2\omega^{-\frac{1}{\varepsilon}}s_{01}}{s_{00} + 2s_{10}}, \quad \frac{H_1}{N_1} = \frac{1-N_1}{N_1} = \eta\frac{(1+\phi^2)s_{11} + \omega^{\frac{1}{\varepsilon}}s_{10}}{(1+\phi^2)s_{11} + s_{01}},$$

where η is given by (22). Solving for N_0 and N_1, and using (7), we arrive at expressions (19) – (20) for the land-use patterns.

Wages and land rents. The ratios W_i/R_i, $i = 0{,}1$, are pinned down by combining (A.1) and (A.2) with the land-market clearing conditions (10):

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha}\left[(1+\eta)s_{00} + 2s_{10} + 2\eta w^{-1}s_{01}\right]^{-1};$$

$$\frac{W_1}{R_1} = \frac{\alpha}{1-\alpha}\left[(1+\phi^2)(1+\eta)s_{11} + s_{01} + \eta w s_{10}\right]^{-1}.$$

Restating the zero-profit conditions (14) as

$$R_i = \alpha^{\alpha}(1-\alpha)^{1-\alpha}A_i\left(\frac{W_i}{R_i}\right)^{-\alpha},$$

and plugging the ratios W_i/R_i into the right-hand sides, we obtain land rents as functions of r, w, and s_{ij}:

$$R_i = \begin{cases}(1-\alpha)A_0\left[(1+\eta)s_{00} + 2s_{10} + 2\eta w^{-1}s_{01}\right]^{\alpha}, & i = 0;\\ (1-\alpha)A_1\left[(1+\phi^2)(1+\eta)s_{11} + s_{01} + \eta w s_{10}\right]^{\alpha}, & i = 1.\end{cases}$$

Plugging the land rents back into the ratios W_i/R_i, we obtain the wages as functions of r, w, and s_{ij}:

$$W_i = \begin{cases}\alpha A_0\left[(1+\eta)s_{00} + 2s_{10} + 2\eta w^{-1}s_{01}\right]^{-(1-\alpha)}, & i = 0;\\ \alpha A_1\left[(1+\phi^2)(1+\eta)s_{11} + s_{01} + \eta w s_{10}\right]^{-(1-\alpha)}, & i = 1.\end{cases}$$

Last, equation (4) may be rewritten as

$$\frac{\left[W_j/(\tau_{ij}R_i^{\beta})\right]^{\varepsilon}}{\sum_{r\in\mathcal{I}}\sum_{s\in\mathcal{I}}\left[W_s/(\tau_{rs}R_r^{\beta})\right]^{\varepsilon}} = \frac{\left[W_j/(\tau_{ij}R_i^{\beta})\right]^{\varepsilon}}{\left[W_0/(R_0^{\beta})\right]^{\varepsilon} + 2\phi\left[W_1/R_0^{\beta}\right]^{\varepsilon} + 2\phi\left[W_0/R_1^{\beta}\right]^{\varepsilon} + 2(1+\phi^2)\left[W_1/(R_1^{\beta})\right]^{\varepsilon}}$$

whose denominator is equal to

$$\left[w/(r^{\beta})\right]^{\varepsilon} + 2\phi\left[1/r^{\beta}\right]^{\varepsilon} + 2\phi\left[w\right]^{\varepsilon} + 2(1+\phi^2) = \omega\rho + 2\phi\rho + 2\phi\omega + 2(1+\phi^2).$$

Q.E.D.

B. Proof of Proposition 1

Using (7) and (A.1) – (A.2), we can define the relative demand for land, $\lambda(r,w)$, as follows:

$$\frac{N_0 + H_0}{N_1 + H_1} = \lambda(r,w) \equiv \frac{(1+\eta)w^{1+\varepsilon}r^{-\beta\varepsilon} + 2\phi w^{1+\varepsilon} + 2\eta\phi r^{-\beta\varepsilon}}{\phi r^{-\beta\varepsilon} + \eta\phi w^{1+\varepsilon} + (1+\phi^2)(1+\eta)}\frac{1}{r}. \tag{B.1}$$

Because each location has one unit of land, the relative supply of land is equal to one. In equilibrium, the relative demand for land equals the relative supply of land: $\lambda(r,w) = 1$. Using (B.1) and (6), the condition $\lambda(r,w) = 1$ becomes:

$$\frac{(1+\eta)\omega^{\frac{1+\varepsilon}{\varepsilon}}\rho + 2\phi\omega^{\frac{1+\varepsilon}{\varepsilon}} + 2\eta\phi\rho}{\phi\rho + \eta\phi\omega^{\frac{1+\varepsilon}{\varepsilon}} + (1+\phi^2)(1+\eta)} = \rho^{-\frac{1}{\beta\varepsilon}},$$

whose solution in $\omega^{\frac{1+\varepsilon}{\varepsilon}}$ yields (23).

To derive (24), let us restate (15), using (6) and (21), as follows

$$\omega^{\frac{\alpha}{\varepsilon}}\rho^{-\frac{1-\alpha}{\beta\varepsilon}} = a\left(\omega\frac{\rho+2\phi}{1+\phi\rho+\phi^2}\right), \tag{B.2}$$

where $a(\cdot)$ is the relative TFP given by (16). Equation (B.2) defines implicitly a function $\omega^{\frac{1+\varepsilon}{\varepsilon}} = g(\rho,\gamma,\delta)$. If $\delta > 0$, then the g-function cannot be expressed in closed form but can be written as a fixed point given by the second line of (24). When $\delta = 0$, the g-function can be expressed in closed form. Indeed, in this case, (B.2) takes the form

$$\omega^{\frac{\alpha}{\varepsilon}}\rho^{-\frac{1-\alpha}{\beta\varepsilon}} = c\left(\omega\frac{\rho+2\phi}{1+\phi\rho+\phi^2}\right)^{\gamma},$$

whose solution in $\omega^{\frac{1+\varepsilon}{\varepsilon}}$ delivers the first line of (24). Q.E.D.

C. Lemma 2

Lemma 2. *The f-function in the RHS of (23) has the following properties:*

i. *there exist $\rho_m > 0$and $\rho_M > \rho_m$, such that $f(\rho) > 0$if and only if $\rho_m < \rho < \rho_M$;*

ii. *$f(\rho)$decreases from $\infty$to 0 over (ρ_m,ρ_M).*

Proof. The proof follows directly from the properties of the relative demand for land, $\lambda(r,w)$, given by (B.1).

The relative demand for land decreases with the relative land price r. Indeed, computing the elasticity of the relative demand for land w.r.t. r, we get:

$$-\frac{\partial \ln\lambda(r,w)}{\partial \ln r} =$$
$$1 + \frac{(1+\eta)\beta\varepsilon\left[\left(1-\phi^2+(1+2\phi^2)\eta\right)w^{1+\varepsilon} + \eta\phi\left(w^{1+\varepsilon}\right)^2 + 2\eta\phi(1+\phi^2)\right]r^{-\beta\varepsilon}}{r\lambda(r,w)\left[\phi r^{-\beta\varepsilon} + \eta\phi w^{1+\varepsilon} + (1+\phi^2)(1+\eta)\right]^2},$$

where the RHS is clearly positive. Also, the relative demand for land increases with the relative wage w. Computing the elasticity of $\lambda(r,w)$ w.r.t. the relative wage w, we get:

$$\frac{\partial \ln \lambda(r,w)}{\partial \ln w} = \frac{(1+\eta)(1+\varepsilon)\left[\phi r^{-2\beta\varepsilon} + \left(1+3\phi^2+\eta(1-\phi^2)\right) r^{-\beta\varepsilon} + 2\phi(1+\phi^2)\right] w^{1+\varepsilon}}{r\lambda(r,w)\left[\phi r^{-\beta\varepsilon} + \eta\phi w^{1+\varepsilon} + (1+\phi^2)(1+\eta)\right]^2},$$

where the RHS is clearly positive. This result reflects two effects: (i) a higher wage leads to substituting labor with land in production; (ii) the citizens, who are commuting-averse, tend to live in locations offering higher wages.

To derive ρ_m and ρ_M, consider two extreme cases.

Extreme case 1: $w = 0$. The condition $\lambda(r,w) = 1$ becomes:

$$\lambda(r,0) \equiv \frac{2\eta\phi r^{-\beta\varepsilon}}{\phi r^{-\beta\varepsilon} + (1+\phi^2)(1+\eta)} \frac{1}{r} = 1. \tag{C.1}$$

The equation $\lambda(r,0) = 1$ has a unique solution $\underline{r} > 0$.

Extreme case 2: $w = \infty$. The condition $\lambda(r,w) = 1$ becomes:

$$\lambda(r,\infty) \equiv \left(\frac{1+\eta}{\eta\phi} r^{-\beta\varepsilon} + \frac{2}{\eta}\right)\frac{1}{r} = 1. \tag{C.2}$$

The equation $\lambda(r,\infty) = 1$ has a unique solution $\overline{r} > \underline{r} > 0$. That $\overline{r} > \underline{r}$ follows from $\frac{\partial \ln \lambda(r,w)}{\partial \ln w} > 0$, which implies $\lambda(r,\infty) > \lambda(r,0)$ for every given r, hence $\lambda(\overline{r},\infty) = 1 = \lambda(\underline{r},0) < \lambda(\underline{r},\infty)$, which implies $\overline{r} > \underline{r}$.

The above analysis brings us to two important conclusions:

- the equilibrium condition $\lambda(r,w) = 1$ defines an increasing relation between r and w, hence it defines a decreasing relation between ω and ρ;
- while the equilibrium condition $\lambda(r,w) = 1$ can hold for any $w \geq 0$ (including $w = 0$ and $w = +\infty$), it can hold only for a limited range of relative land rents: $r \in [\underline{r},\overline{r}]$.

Because $\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho)$ is just an equivalent way of writing the equilibrium condition $\lambda(r,w) = 1$, which defines a decreasing relationship between ρ and ω, we have $f'(\rho) < 0$ for each admissible value of ρ. Furthermore, because the equilibrium condition $\lambda(r,w) = 1$ can only hold for $r \in [\underline{r},\overline{r}]$, the extreme cases of $r = \underline{r}$ and $r = \overline{r}$, which correspond, respectively, to $w = 0$ and $w = \infty$, the function f decreases from ∞ to 0 as ρ changes from $\rho_m \equiv \overline{r}^{-\beta\varepsilon}$ to $\rho_M \equiv \underline{r}^{-\beta\varepsilon} > \rho_m$. Beyond the interval (ρ_m,ρ_M), the expression for $f(\rho)$ in (23), although still mathematically well defined, has no economic meaning. Q.E.D.

D. Lemma 3

Lemma 3. *(i) If $\gamma \neq \gamma_m$ and $\delta = 0$, then $g(\rho;\gamma)$is strictly positive and finite over $[\rho_m,\rho_M]$. (ii) If $\gamma < \gamma_m$, then gis increasing over $[\rho_m,\rho_M]$. (iii) If $\gamma > \gamma_m$, then gis decreasing over $[\rho_m,\rho_M]$. (iv) As γ converges to γ_m, we have:*

$$\lim_{\gamma\nearrow\gamma_m} g(\rho;\gamma) = \begin{cases} 0, & \rho < \rho_L; \\ \left(\frac{\rho_L+2\phi}{1+\phi\rho_L+\phi^2}\right)^{-\frac{1+\varepsilon}{\varepsilon}} & \rho = \rho_L; \\ \infty, & \rho > \rho_L; \end{cases} \qquad \lim_{\gamma\searrow\gamma_m} g(\rho;\gamma) = \begin{cases} \infty, & \rho < \rho_L; \\ \left(\frac{\rho_L+2\phi}{1+\phi\rho_L+\phi^2}\right)^{-\frac{1+\varepsilon}{\varepsilon}} & \rho = \rho_L; \\ 0, & \rho > \rho_L; \end{cases}$$

where $\rho_L > 0$is the unique solution to the equation

$$\rho^{\frac{1}{\eta}} \frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} = c^{\frac{-\varepsilon}{\alpha}}.$$

Proof. Part (i) follows from combining (24) with $0 < \rho_m < \rho_M < \infty$. Parts (ii) and (iii) are obtained by differentiating g with respect to ρ. Part (iv) holds because $g(\rho;\gamma)$ may be rewritten as follows:

$$g(\rho;\gamma) = \Phi \left[\left(\rho^{\frac{1}{\eta}} \frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} \right)^{\frac{\gamma_m}{\gamma_m - \gamma}} \cdot \left(\frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} \right)^{-1} \right]^{\frac{1+\varepsilon}{\varepsilon}}, \tag{D.1}$$

where $\Phi \equiv c^{\frac{\psi\eta}{\gamma_m - \gamma}}$. Q.E.D.

E. Lemma 4

Lemma 4. *There exists a function $\overline{\phi}(\beta\varepsilon) \in (0,1)$and scalar $\overline{\eta} > 0$such that $\rho_m < \rho_L$if $\phi < \overline{\phi}$or $\eta < \overline{\eta}$. Conversely, if $\phi > \overline{\phi}$and $\eta > \overline{\eta}$, then $\rho_m > \rho_L$.*

Proof. It follows from the proof of Lemma 2 that ρ_m is the unique solution of

$$D(\rho) \equiv (1 + \eta)\rho^{\frac{1+\beta\varepsilon}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - \eta\phi = 0. \tag{E.1}$$

The expressions (D.1) and (E.1) imply that ρ_L and ρ_m are functions of η. We next show that ρ_m and ρ_L vary with η as follows,

$$\lim_{\eta\to 0} \rho_L = 1, \quad \frac{d\rho_L}{d\eta} < 0, \quad \lim_{\eta\to\infty} \rho_L = 1 - \phi,$$

$$\lim_{\eta\to 0}\rho_m = 0, \quad \frac{d\rho_m}{d\eta} > 0, \quad \lim_{\eta\to\infty}\rho_m = \phi^{\frac{\beta\varepsilon}{1+\beta\varepsilon}}.$$

We can show that ρ_m (resp., ρ_L) increases (resp., decreases) in η by applying the implicit function theorem to $D(\rho) = 0$ (resp., (D.1)). Observe further that, when $\eta \to \infty$ (resp., $\eta \to 0$), dividing $D(\rho) = 0$ by η and taking the limit yields $\rho_m = \phi^{\beta\varepsilon/(1+\beta\varepsilon)}$ (resp., $\rho_m = 1$). Last, when $\eta \to \infty$ (resp., $\eta \to 0$), taking (E.1) at the power η and the limit yields $\rho_L = 1 - \phi$ (resp., $\rho_L = 1$).

To determine where ρ_m and ρ_L intersect, we compare $\lim_{\eta\to\infty}\rho_L$ and $\lim_{\eta\to\infty}\rho_m$ by considering the equation

$$\phi^{\beta\varepsilon/(1+\beta\varepsilon)} + \phi = 1. \tag{E.2}$$

Differentiating the LHS of (E.2) with respect to ϕ shows that it increases from 0 to 2 when ϕ increases from 0 to 1. The intermediate value theorem then implies that, for any given $\beta\varepsilon$, (E.2) has a unique solution $\overline{\phi}(\beta\varepsilon) \in (0,1)$, which increases with $\beta\varepsilon$.

The inequality $\rho_m \leq \rho_L$ holds if $\phi^{\beta\varepsilon/(1+\beta\varepsilon)} \leq 1 - \phi$, which amounts to $\phi \leq \overline{\phi}(\beta\varepsilon)$. If $\overline{\phi} < \phi \leq 1$, then there exists a unique value $\overline{\eta} > 0$ that solves the condition $\rho_L(\eta) = \rho_m(\eta)$. Consequently, if $\eta < \overline{\eta}$, then $\rho_m \leq \rho_L$. If $\eta \geq \overline{\eta}$, then $\rho_m > \rho_L$.

Summing up, $\rho_m \leq \rho_L$ if $\phi \leq \overline{\phi}$ or $\eta \leq \overline{\eta}$, and $\rho_m > \rho_L$ when both conditions fail. Q.E.D.

F. Iterative stability

One candidate, particularly relevant for the literature on quantitative spatial models, is to say that an equilibrium is stable if an iterative process will converge to it. Formally, if equilibria are defined by $f(\rho) = g(\rho)$, then equilibria are fixed points of $h(\rho) = \rho$, for $h(\rho) \equiv f^{-1}(g(\rho))$. In this case, it is well known that an iterative process will find a fixed point ρ^* if and only if $|h'(\rho^*)| < 1$. Surprisingly, this notion of stability is not well defined as there are two problems.

First, there are two alternative ways of stating the fixed point problem. First, as $\rho = h(\rho) \equiv f^{-1}(g(\rho))$, and alternatively, as $\rho = \widetilde{h}(\rho) \equiv g^{-1}(h(\rho))$. Both representations have the same solutions, but their stability properties are opposite. Indeed, for any solution of this problem, $|h'(\rho^*)| < 1$ if and only if $|\widetilde{h}'(\rho^*)| > 1$. Thus, the iterative stability of any given solution to the fixed point problem that defines equilibrium depends on arbitrary decisions about the representation of the fixed point problem.

To understand the second problem, observe that for any $0 < \theta < 1$, the equation $h(\rho) = \theta\rho + (1-\theta)\rho$ also defines solutions of $f(\rho) = g(\rho)$, so that fixed points of $\widetilde{h}(\rho) = [(h(\rho) - (1-\theta)\rho)/\theta] = \rho$ are also solutions of $f(\rho) = g(\rho)$. However, the stability properties of this second equation may be different from the original. By choosing θ sufficiently small, we guarantee that $|\widetilde{h}'(\rho^*)| > 1$.

In sum, iterative stability is not well defined and iterative methods cannot be expected to find all the equilibria of an economy when multiplicity of equilibria prevails.

An alternative approach to stability involves specifying state variables for the economy and adjustment process for these state variables. In the context of our problem, symmetry implies that we must determine the values of three variables to obtain the equilibrium outcome. For example, it is sufficient to know L_0, M_0, and s_{00} to determine all the s_{ij}. To implement this notion of stability, we must specify an adjustment process describing how L_0, M_0 and s_{00} respond to a perturbation. Stability is then well defined in the resulting dynamic system. However, this approach rests on ad hoc descriptions of the adjustment process, which is hard to justify here.

G. Lemma 5

Lemma 5. *For any two distinct location pairs $ij$and klsuch that $s^*_{ij} > 0$, there exists an individual $\nu \in [0,1]$with $z_{ij}(\nu) \in S_{ij}$and $z_{kl}(\nu) > 0$who is indifferent between ij and kl.*

Proof. The assumption $s^*_{ij} > 0$ implies $L^*_j > 0$, hence $W^*_j > 0$. Combining this with (3) and (26) implies that any individual $\nu \in [0,1]$ whose type $\mathbf{z}(\nu)$ satisfies

$$z_{ij}(\nu) = z_{kl}(\nu)\frac{W^*_l}{W^*_j}\frac{\tau_{kl}}{\tau_{ij}}\left(\frac{R^*_i}{R^*_k}\right)^{\beta} \geq z_{od}(\nu)\frac{W^*_d}{W^*_j}\frac{\tau_{od}}{\tau_{ij}}\left(\frac{R^*_o}{R^*_i}\right)^{\beta} \tag{G.1}$$

is indifferent between ij and kl.

Two cases may arise. First, if $s^*_{kl} > 0$, then $L^*_l > 0$ and $W^*_l > 0$. (G.1) thus implies that any individual ν satisfying

$$z_{kl}(\nu) > 0, \quad z_{ij}(\nu) = z_{kl}(\nu)\frac{W^*_l}{W^*_j}\frac{\tau_{kl}}{\tau_{ij}}\left(\frac{R^*_i}{R^*_k}\right)^{\beta}, \quad z_{od}(\nu) = 0$$

is indifferent between ij and kl.

Second, if $s^*_{kl} = 0$, then $L^*_l = 0$ and $W^*_l = 0$. Therefore, (G.1) implies that any individual such that $z_{kl}(\nu) > 0$ and $z_{ij}(\nu) = 0$ for any $ij \neq kl$ is indifferent between ij and kl. Q.E.D.

H. Existence and uniqueness of a conditional equilibrium price system

Step 1. We first show the existence of a unique conditional equilibrium price for a symmetric commuting pattern $\mathbf{s}$ such that either $\mathbf{L}(\mathbf{s}) = (0,1,0)$ or $\mathbf{L}(\mathbf{s}) = (1/2,0,1/2)$, and $M_i(\mathbf{s}) > 0$ for $i = 0, \pm 1$.

We focus on the case of a fully agglomerated labor supply pattern, i.e., such that $L_0(\mathbf{s}) = 1$ and $L_{-1}(\mathbf{s}) = L_1(\mathbf{s}) = 0$ (the proof for the fully dispersed labor supply pattern given by $L_0 = 0$ and $L_{-1}(\mathbf{s}) = L_1(\mathbf{s}) = 1/2$ goes along the same lines). Plugging $L_0 = 1$ and $L_{-1} = L_1 = 0$ into the firm's complementary slackness conditions at $i = 0$, we obtain

$$W_0 = \alpha N_0^{1-\alpha} \qquad \text{and} \qquad R_0 = (1-\alpha)N_0^{-\alpha}, \tag{H.1}$$

so that

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha} N_0. \tag{H.2}$$

Observe that $L_1(\mathbf{s}) = L_{-1}(\mathbf{s}) = 0$ implies $s_{i1} = s_{i,-1} = 0$ for all $i \in \{-1,0,1\}$. Combining this with the land market clearing condition and the market residential demand at $i = 0$, we get:

$$H_0 + N_0 = 1 \qquad \text{and} \qquad H_0 = s_{00}\frac{W_0}{R_0},$$

so that

$$N_0 = 1 - H_0 = 1 - s_{00}\frac{W_0}{R_0}. \tag{H.3}$$

Plugging (H.3) into (H.2), we get a linear equation in W_0/R_0:

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha}\left(1 - s_{00}\frac{W_0}{R_0}\right) \quad \Longrightarrow \quad \frac{\overline{W}_0(\mathbf{s})}{\overline{R}_0(\mathbf{s})} = \frac{\alpha}{1-\alpha+\alpha s_{00}}. \tag{H.4}$$

From (H3)-(H.4), we get:

$$\overline{N}_0(\mathbf{s}) = \frac{1-\alpha}{1-\alpha+\alpha s_{00}}.$$

Plugging $N_0 = \overline{N}_0(\mathbf{s})$ into the equilibrium condition (H.1) pins down uniquely the conditional equilibrium wage $\overline{W}_0(\mathbf{s})$ and the conditional equilibrium land rent $\overline{R}_0(\mathbf{s})$. As for $\overline{W}_j(\mathbf{s})$ and $\overline{R}_i(\mathbf{s})$ for $i,j = \pm 1$, zero labor supply implies $\overline{W}_j(\mathbf{s}) = 0$ and $\overline{N}_j(\mathbf{s}) = 0$ for $j = \pm 1$. Hence, the land market clearing at the periphery becomes

$$H_i = 1 = s_{i0}\frac{W_0}{R_i} \qquad \text{for} \qquad i = \pm 1,$$

which implies $\overline{R}_i(\mathbf{s}) = s_{i0}\overline{W}_0(\mathbf{s})$ for $i = \pm 1$.

I. Proof of Proposition 2

Plugging $b = c = d = 1$ and $\gamma = \delta = 0$ into (23), we get

$$f(\rho) \equiv \frac{\phi\rho - 2\eta\phi\rho^{1+\frac{1}{\beta\varepsilon}} + (1+\phi^2)(1+\eta)}{(1+\eta)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - \eta\phi}, \qquad g(\rho) = \rho^{\psi},$$

where

$$\psi \equiv \frac{(1-\alpha)(1+\varepsilon)}{\alpha\beta\varepsilon} = \frac{1+\varepsilon}{\eta\varepsilon}.$$

(i) Because $f(\rho)$ decreases with ρ from ∞ to 0 over (ρ_m, ρ_M), and $g(\rho)$ increases with ρ from 0 to ∞, the two curves have a unique intersection $\rho^* \in (\rho_m, \rho_M)$, which implies existence and uniqueness of equilibrium and that it is interior.

(ii) It is readily verified that $0 < f(1) < g(1) = 1$. Hence, the intersection must occur strictly between $\rho_m < 1$ and 1. This implies $0 < \rho^* < 1$ and $\omega^* = (\rho^*)^{\frac{1}{\eta}} < 1$.

(iii) The equilibrium employment pattern is bell-shaped if and only if

$$\ell^* = \omega^* \frac{\rho^* + 2\phi}{1 + \phi\rho^* + \phi^2} = (\rho^*)^{\frac{1}{\eta}} \frac{\rho^* + 2\phi}{1 + \phi\rho^* + \phi^2} > 1.$$

Restate the equilibrium condition $f(\rho) = g(\rho)$ as follows:

$$\frac{\left(\frac{\eta}{1+\eta}\rho^{-\psi} + \frac{1}{1+\eta}\rho^{-1}\right)^{-1} + 2\phi}{\phi\left(\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho\right) + 1 + \phi^2} \left(\frac{\eta}{1+\eta}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\rho^{\psi+\frac{1}{\beta\varepsilon}}\right) = 1. \tag{I.1}$$

Because $1/x$ is convex, for every $\rho < 1$ Jensen's inequality implies

$$\left(\frac{\eta}{1+\eta}\rho^{-\psi} + \frac{1}{1+\eta}\rho^{-1}\right)^{-1} < \frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho < \rho. \tag{I.2}$$

Plugging (I.2) into (I.1) leads to

$$1 < \frac{\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho + 2\phi}{\phi\left(\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho\right) + 1 + \phi^2} \left(\frac{\eta}{1+\eta}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\rho^{\psi+\frac{1}{\beta\varepsilon}}\right).$$

Using $\psi > 1$ yields

$$\frac{\eta}{1+\eta}\rho^{\psi} + \frac{1}{1+\eta}\rho < \frac{\eta}{1+\eta}\rho + \frac{1}{1+\eta}\rho = \rho.$$

Because the function $\frac{x+2\phi}{\phi x + 1 + \phi^2}$ is increasing for all $x \geq 0$, we obtain

$$1 < \frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2} \left(\frac{\eta}{1+\eta}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\rho^{\psi+\frac{1}{\beta\varepsilon}}\right). \tag{I.3}$$

As $\psi > 1$ implies

$$\frac{1}{\eta} < 1 + \frac{1}{\beta\varepsilon} < \psi + \frac{1}{\beta\varepsilon},$$

while $\rho^* < 1$, we have

$$\frac{\eta}{1+\eta}\left(\rho^*\right)^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+\eta}\left(\rho^*\right)^{\psi+\frac{1}{\beta\varepsilon}} < \left(\rho^*\right)^{\frac{1}{\eta}}.$$

Replacing the bracketed term in (I.3), we obtain the inequality:

$$1 < \left(\rho^*\right)^{\frac{1}{\eta}} \frac{\rho^* + 2\phi}{\phi\rho^* + 1 + \phi^2},$$

which is equivalent to $\rho^* > \rho_L$, hence $\ell^* > 1$ (see (7)).

(iv) When $\varepsilon \to \infty$, we have:

$$\lim_{\varepsilon\to\infty} f(\rho) = \rho^{-1},\ \lim_{\varepsilon\to\infty} g(\rho) = \rho^{\frac{1}{\eta}} \implies \lim_{\varepsilon\to\infty} \rho^* = \lim_{\varepsilon\to\infty} \omega^* = 1.$$

Also, $\lim_{\varepsilon\to\infty} \phi = 0$. Hence, setting $\phi = 0$ in the RHS of (7), we get:

$$\lim_{\varepsilon\to\infty} \begin{pmatrix} s_{11} & s_{10} & s_{1-1} \\ s_{01} & s_{00} & s_{0-1} \\ s_{-11} & s_{-10} & s_{-1-1} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Q.E.D.

J. Lemma 6 and proof of Proposition 3

We first prove the following lemma.

Lemma 6. *Consider an interior equilibrium (ω^*,ρ^*), such that $g'(\rho^*) > f'(\rho^*)$. Any shock in c, γor δthat shifts the g-curve upwards/downwards in the vicinity of the equilibrium leads to a labor pattern more/less concentrated at the center.*

Proof. Combining the labor centrality ratio (21) with (23), we get:

$$\ell^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho)\left(\frac{\rho + 2\phi}{1 + \phi\rho + \phi^2}\right)^{\frac{1+\varepsilon}{\varepsilon}}. \tag{J.1}$$

It is readily verified that the right-hand side of (J.1) decreases in ρ. Because f is independent of c, the right-hand side of (J.1) as a function of ρ is also independent of c. Note, however, that the equilibrium value of ρ depends on c, which implies that the equilibrium value of ℓ varies with c. Indeed, an upward/downward shift in the g-curve leads to a decrease/increase in ρ^* because the equilibrium moves northwestwards/southeastwards along the f-curve which is unaffected by the change in the value of c.

Hence, we have:

$$\text{an upward shift in } g \Longrightarrow \rho^* \downarrow \Longrightarrow \ell^* \uparrow .$$

Q.E.D.

From (23) – (24), one can see that an increase in c keeps the f-curve unchanged and shifts upwards the g-curve. Hence, by Lemma 6, we have:

$$\frac{d\ell^*}{dc} > 0. \tag{J.2}$$

In other words, ℓ^* is strictly increasing in c.

We now prove (i) – (v) in the following order: (i), (v), (iii), (ii) and (iv).

(i) We need to show that

$$\lim_{c\to 0} \ell^* = 0.$$

Because $\lim_{c\to 0} g(\rho) = 0$ for $\forall\, \rho \in (\rho_m, \rho_M)$, we have:

$$\begin{cases} \lim\limits_{c\to 0} \omega^*(c) = 0 \\ \lim\limits_{c\to 0} \rho^*(c) = \rho_M \end{cases} \Longrightarrow \qquad \lim_{c\to 0} \ell^*(c) = \lim_{c\to 0} \left[\frac{\rho^*(c) + 2\phi}{1 + \phi\rho^*(c) + \phi^2} \omega^*(c) \right] = 0. \tag{J.3}$$

(v) Along the same lines as in the proof of (i), one can show that

$$\begin{cases} \lim\limits_{c\to \infty} \omega^*(c) = \infty \\ \lim\limits_{c\to \infty} \rho^*(c) = \rho_m \end{cases} \Longrightarrow \qquad \lim_{c\to \infty} \ell^*(c) = \lim_{c\to \infty} \left[\frac{\rho^*(c) + 2\phi}{1 + \phi\rho^*(c) + \phi^2} \omega^*(c) \right] = \infty. \tag{J.4}$$

(iii) Using the implicit function theorem shows that $\ell^*(c)$ is differentiable, hence continuous, w.r.t. c for $0 < c < \infty$. Combining this with (J.2) – (J.4) implies that the equation $\ell^*(c) = 1$ has a unique, finite, and positive, solution $\bar{c}$.

(ii) and (iv) follow from (J.1) combined with (i), (iii), and (v). Q.E.D.

K. Proof of Proposition 4

(i) Assume that $\delta = 0$. Using (16) implies

$$a(\gamma) = c\ell^\gamma$$

$$E(\gamma) = C_0 \left(\frac{\ell}{\ell+2}\right)^{\gamma}, \qquad F(\gamma) = C_1 \left(\frac{1}{\ell+2}\right)^{\gamma} \Longrightarrow a(\gamma) = \frac{E(\gamma)}{F(\gamma)} = \frac{C_0}{C_1}\ell^{\gamma}.$$

Therefore,

$$\left.\frac{da(\gamma)}{d\gamma}\right|_{\gamma=0} = c \ln \ell.$$

Proposition 4 implies that $\widehat{c} > 0$ exists such that $\ell(\widehat{c}) = 1$. Therefore, increasing returns magnifies the initial advantage of a location given by the value of c when $c > \widehat{c}$ because raising γ above 0 increases $a(\gamma)$. The opposite holds when $c < \widehat{c}$. Furthermore, the intensity of the effect of increasing returns increases with the value of $c > \widehat{c}$.

(ii) Assume that $\gamma = 0$. In this case, we have

$$a(\delta) = c\frac{1+2\delta}{\delta+1+\delta^2}.$$

Setting $\gamma = 0$ in (24), we obtain the following closed-form solution for function g:

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} = c^{\frac{1+\varepsilon}{\alpha}} \left(\frac{1+2\delta}{1+\delta+\delta^2}\right)^{\frac{1+\varepsilon}{\alpha}} \rho^{\psi}.$$

Since the term between parentheses is larger than 1 for all $\delta > 0$, an increase in δ from $\delta = 0$ to $\delta = (\sqrt{3}-1)/2 \simeq 0.366\,03$ shifts function g upwards. As function f remains unaffected, it follows from Lemma 6 in Appendix J that ℓ^* increases. On the other hand, when δincreases over $((\sqrt{3}-1)/2{,}1)$, ℓ^* decreases.

L. Proof of Proposition 5

(i) When $\delta = 0$ and $\gamma \neq \gamma_m$, from Lemmas 2 and 3, $f(\rho) > g(\rho{,}\gamma)$ when ρ slightly exceeds ρ_m, while the opposite inequality holds when ρ is close enough to ρ_M. Hence, by relocating a small subset of individuals from ij to kl, the commuting pattern $\mathbf{s}$ becomes different from the equilibrium pattern $\mathbf{s}^*$. Hence, given our definition of stability, we compare the equilibrium and off-equilibrium utility levels. For this to be possible, we must determine the conditional equilibrium vectors of wages and land rents $\overline{\mathbf{W}}(\mathbf{s})$ and $\overline{\mathbf{R}}(\mathbf{s})$. We show in Appendix H that, for $\alpha > 1/2$, these vectors exist, are unique and continuous in $\mathbf{s}$. Hence, by the intermediate value theorem, the equilibrium condition $f(\rho) = g(\rho{,}\gamma)$ has an interior solution $\rho^* \in (\rho_m{,}\rho_M)$.

(ii) We show the existence and unstability of the two corner equilibria. Consider first the wage pattern $W_0 = 0 < W_1$, hence $\omega = w = 0$. The utility-maximizing commuting flows (7) imply that, at the central location $i = 0$, labor supply $= 0 \Longrightarrow A_0 = 0 \Longrightarrow$ labor demand $= 0$. The land-market clearing condition $\lambda(r,w) = 1$ takes the form of (C.1), which means $r^* = \underline{r}$, i.e., $\rho^* = \rho_M$.

Assume now that $L_0^* = 1$ (the proof for $L_{-1}^* = L_1^* = 1/2$ goes along the same lines). Consider an individual ν such that, for all $i \in \{-1,0,1\}$, ν's match values satisfy $z_{ij}(\nu) = 0$ for $j = 0, \pm 1$. Clearly, ν is indifferent between working at the center and working at the periphery (in both cases, she enjoys zero utility). Consider a positive-measure set of individuals whose tastes are close to those of ν and whose utility-maximizing choice is $ij = 00$. Relocating them (together with ν) from $ij = 00$ to $kl = 01$, we have $V_{01}(\nu,\mathbf{s}) > 0$ because $\overline{W}_1(\mathbf{s}) > 0$. Using the complementary slackness condition $(\alpha A_j L_j^{\alpha-1} N_j^{1-\alpha} - W_j)L_j = 0$, there exists a positive-measure subset of individuals who are strictly better-off working at location $j = 1$. As a result, the corner equilibrium $L_0^* = 1$ is an unstable equilibrium.

(iii) Last, we show that $M_i^* > 0$ for all i. Assume that $R_i^* = 0$ at i. Because there is a location j such that $W_j^* > 0$, households who choose the pair ij enjoys an infinite utility level, which implies $s_{ij} > 0$. These households' land demand is thus infinite while the land supply is finite, a contradiction. Q.E.D.

M. Lemma 7

Lemma 7. *There exists a function $F(\rho)$ independent of γsuch that an interior equilibrium $\rho^*$is stable if and only if $F(\rho^*) > 1$. This function is continuous over $(\rho_m,\rho_{CR})$and over (ρ_{CR},ρ_M), satisfies $F(\rho_m) = F(\rho_M) = 0$, and has a vertical asymptote at $\rho = \rho_{CR}$.*

Proof: The proof involves four steps.

Step 1. We first show the existence of a unique conditional equilibrium price for a symmetric commuting **s** such that $s_{ij} > 0$ for all i,j when $\alpha > 1/2$.

Because $L_i > 0$ for $i = 0, \pm 1$, the first-order conditions for the production sector yields the equilibrium conditions:

$$W_j = \alpha A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha}, \tag{M.1}$$

$$R_j = (1-\alpha) A_j \left(\frac{L_j}{N_j}\right)^{\alpha}. \tag{M.2}$$

Furthermore, we also know that housing market clearing at location i yields:

$$H_i = \frac{\beta}{R_i} \sum_{j=1}^{n} s_{ij} W_j. \tag{M.3}$$

Plugging (M.1) and (M.2) into (M.3), and using the land market balance condition $N_i + H_i = 1$, we get:

$$H_i = 1 - N_i = \frac{\alpha\beta}{(1-\alpha)A_i} \left(\frac{N_i}{L_i}\right)^{\alpha} \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha},$$

$$(1-\alpha)A_i (1-N_i) \left(\frac{N_i}{L_i}\right)^{-\alpha} = \alpha\beta \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha},$$

$$(1-\alpha)A_i \left(\frac{N_i}{L_i}\right)^{-\alpha} = (1-\alpha)A_i L_i \left(\frac{N_i}{L_i}\right)^{1-\alpha} + \alpha\beta \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha}.$$

Because **s** is symmetric, this system of equations becomes:

$$(1-\alpha)A_0 \left(\frac{N_0}{L_0}\right)^{-\alpha} = \left[(1-\alpha)L_0 + \alpha\beta s_{00}\right] A_0 \left(\frac{N_0}{L_0}\right)^{1-\alpha} + 2\alpha\beta s_{01} A_1 \left(\frac{N_1}{L_1}\right)^{1-\alpha}$$

$$(1-\alpha)A_1 \left(\frac{N_1}{L_1}\right)^{-\alpha} = \alpha\beta s_{10} A_0 \left(\frac{N_0}{L_0}\right)^{1-\alpha} + \left[(1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})\right] A_1 \left(\frac{N_1}{L_1}\right)^{1-\alpha}$$

Dividing one equation by the other and using $A_i = L_i^{\gamma}$ for $i = 0, \pm 1$, we get:

$$n^{-\alpha}\ell^{\gamma+\alpha} = \frac{\left[(1-\alpha)L_0 + \alpha\beta s_{00}\right] \ell^{\gamma} \left(\frac{n}{\ell}\right)^{1-\alpha} + 2\alpha\beta s_{01}}{\alpha\beta s_{10} \ell^{\gamma} \left(\frac{n}{\ell}\right)^{1-\alpha} + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})} \tag{M.4}$$

Because (M.1) and (M.2) imply

$$n^{-\alpha}\ell^{\gamma+\alpha} = r, \qquad \ell^{\gamma} \left(\frac{n}{\ell}\right)^{1-\alpha} = w, \tag{M.5}$$

we have

$$w^{\alpha} r^{1-\alpha} = \ell^{\gamma} = \left(\frac{L_0}{L_1}\right)^{\gamma}. \tag{M.6}$$

Likewise, combining (M.4) and (M.5), we get:

$$r = \frac{[(1-\alpha)L_0 + \alpha\beta s_{00}]\, w + 2\alpha\beta s_{01}}{\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}. \tag{M.7}$$

A sufficient condition for the system (M.6) – (M.7) to have a unique solution $(\overline{w}(\mathbf{s}),\overline{r}(\mathbf{s}))$ is that the graph of the relationship (M.7) between w and r intersects the downward-sloping curve given by (M.6) from below. The RHS of (M.7) is the ratio of two positive linear increasing functions of w. Because the elasticity of a linear increasing function with a positive intercept never exceeds 1, the elasticity of the RHS of (M.7) w.r.t. w is always larger than -1. Restating (M.6) as

$$r = \ell^{\frac{\gamma}{1-\alpha}} w^{-\frac{\alpha}{1-\alpha}}$$

shows that the elasticity of the RHS of this expression w.r.t. w equals $-\alpha/(1-\alpha)$, which is smaller than -1 when $\alpha > 1/2$.

Step 2. Denote by $\left(\overline{\mathbf{W}}(\mathbf{s}),\overline{\mathbf{R}}(\mathbf{s})\right)$ the equilibrium price vector conditional to an arbitrary commuting pattern $\mathbf{s}$ that belongs to a neighborhood of an interior equilibrium commuting pattern $\mathbf{s}^*$, and let $\overline{w}(\mathbf{s})$ and $\overline{r}(\mathbf{s})$ be the corresponding wage ratio and the land-price ratio:

$$\overline{w}(\mathbf{s}) \equiv \frac{\overline{W}_0(\mathbf{s})}{\overline{W}_1(\mathbf{s})} \quad \text{and} \quad \overline{r}(\mathbf{s}) \equiv \frac{\overline{R}_0(\mathbf{s})}{\overline{R}_1(\mathbf{s})}.$$

Consider the following two types of relocations: $0j \to 1j$ (changing place of residence but not the workplace) and $i0 \to i1$ (changing the workplace but not the place of residence). Observe that, in equilibrium, for each individual ν, we have:

$$\frac{V^*_{0j}(\nu)}{V^*_{1j}(\nu)} = \frac{z_{0j}(\nu)}{z_{1j}(\nu)} \left(\overline{r}(\mathbf{s}^*)\right)^{-\beta}, \tag{M.8}$$

$$\frac{V^*_{i0}(\nu)}{V^*_{i1}(\nu)} = \frac{z_{i0}(\nu)}{z_{i1}(\nu)} \overline{w}(\mathbf{s}^*). \tag{M.9}$$

If the individual ν is indifferent between $0j$ and $1j$ for some $j = \{-1,0,1\}$, switching from $0j$ to $1j$ makes this individual strictly worse off if and only if $\overline{r}(\mathbf{s}^*)$ decreases when a small subset of residents (almost indifferent between $0j$ and $1j$) of measure Δ is moved from 0 to 1, i.e.,

$$\frac{\partial \overline{r}(\mathbf{s}^*)}{\partial s_{1j}} - \frac{\partial \overline{r}(\mathbf{s}^*)}{\partial s_{0j}} < 0 \tag{M.10}$$

because (M.8) and (M.10) imply that $V^*_{0j}(\nu)/V^*_{1j}(\nu)$ increases above 1.

Likewise, using (M.9) if ν is an individual indifferent between $i0$ and $i1$ for some $i = \{-1{,}0{,}1\}$, switching from $i0$ to $i1$ makes ν strictly worse off if and only if $\overline{w}(\mathbf{s}^*)$ increases when a small subset of households (almost indifferent between $i0$ and $i1$) of measure Δ is moved from 0 to 1, i.e.,

$$\frac{\partial \overline{w}(\mathbf{s}^*)}{\partial s_{i1}} - \frac{\partial \overline{w}(\mathbf{s}^*)}{\partial s_{i0}} > 0. \qquad \text{(M.11)}$$

Step 3. We now show that the land-price ratio $\overline{r}(\mathbf{s}^*)$ always satisfies the equilibrium condition (M.10). Under a relocation of residents from $0j$ to $1j$ (or, equivalently, from $1j$ to $0j$) for $j = 0{,}1$, the numerator in the RHS of (M.7) decreases pointwise, while the denominator increases pointwise. Therefore, the curve (M.7) shifts downwards in the $(w{,}r)$-plane, while the curve (M.6) remains unchanged. Because (M.7) intersects (M.6) from below, this implies a reduction in $\overline{r}(\mathbf{s})$. Hence, (M.10) holds.

Step 4. It remains to check when (M.11) holds. To this end, we study when the relocation of a $\Delta-$measure subset of households from $i0$ to $i1$ for $i = 0, \pm 1$ leads to an increase in the relative wage $\overline{w}(\mathbf{s})$. As a result, two cases must be distinguished: (i) a relocation of households from 00 to 01 and (ii) a relocation of households from 10 to 11.

Taking the log-differential of (M.6) yields:

$$\alpha d \log w + (1-\alpha) d \log r = \gamma (d \log L_0 - d \log L_1). \qquad \text{(M.12)}$$

Two cases may arise.

(i) Assume that

$$ds_{00} = -\Delta, \qquad ds_{01} = ds_{0,-1} = \Delta/2,$$

$$ds_{ij} = 0 \text{ otherwise.}$$

In this case, (M.12) becomes:

$$\alpha \, d \log w + (1-\alpha) \, d \log r = \gamma \left(\frac{ds_{00}}{L_0} - \frac{ds_{01}}{L_1} \right) = -\gamma \Delta \left(\frac{1}{2L_1} + \frac{1}{L_0} \right)$$

Taking the log-differential of (M.7) yields:

$$d\log r = \frac{d\left[\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}\right]}{\left[(1-\alpha)L_0+\alpha\beta s_{00}\right]w+2\alpha\beta s_{01}} - \frac{d\left[\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})\right]}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}. \tag{M.13}$$

Because

$$d\left[\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}\right] = -\Delta\left(1-\alpha+\alpha\beta\right)w+\alpha\beta\Delta+\left((1-\alpha)L_0+\alpha\beta s_{00}\right)wd\log w,$$

while

$$d\left[\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})\right] = (1-\alpha)\frac{\Delta}{2}+\alpha\beta s_{10}wd\log w,$$

(M.13) becomes

$$d\log r = \left[\frac{-\left(1-\alpha+\alpha\beta\right)w+\alpha\beta}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}} - \frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]\Delta$$

$$+\left[\frac{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]d\log w$$

Plugging this expression into (M.12), we get:

$$d\log w = \frac{-\gamma\left(\frac{1}{2L_1}+\frac{1}{L_0}\right)+(1-\alpha)\left[\frac{(1-\alpha+\alpha\beta)w-\alpha\beta}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}{\alpha+(1-\alpha)\left[\frac{((1-\alpha)L_0+\alpha\beta s_{00})w}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}}-\frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}\Delta.$$

When $\alpha > 1/2$, the denominator in $d\log w$ is always positive because each bracketed term of the denominator is smaller than 1. As a result, the stability condition $d\log w > 0$ holds if the numerator is positive:

$$\frac{\left(1-\alpha+\alpha\beta\right)w-\alpha\beta}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})} > \frac{\gamma}{1-\alpha}\left(\frac{1}{L_0}+\frac{1}{2L_1}\right). \tag{M.14}$$

(ii) We now assume that

$$ds_{11} = -ds_{10} = \Delta/2, \qquad ds_{-10} = -ds_{-1,-1} = -\Delta/2,$$

$$ds_{ij} = 0 \text{ otherwise.}$$

Hence, (M.12) becomes:

$$\alpha d\log w + (1-\alpha)\, d\log r = \gamma \left[\frac{ds_{10} + ds_{-10}}{L_0} - \frac{ds_{11}}{L_1} \right] = -\gamma\Delta \left(\frac{1}{2L_1} + \frac{1}{L_0} \right)$$

Because

$$d\left[\left((1-\alpha)L_0 + \alpha\beta s_{00} \right) w + 2\alpha\beta s_{01} \right] = -\Delta\,(1-\alpha)\, w + \left((1-\alpha)L_0 + \alpha\beta s_{00} \right) w d\log w$$

and

$$d\left[\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1}) \right] = \alpha\beta \frac{\Delta}{2} w + (1-\alpha)\frac{\Delta}{2} + \alpha\beta s_{10} w d\log w,$$

(M.13) becomes

$$d\log r = \left[\frac{-\,(1-\alpha)\, w}{\left((1-\alpha)L_0 + \alpha\beta s_{00} \right) w + 2\alpha\beta s_{01}} - \frac{1}{2} \frac{1 - \alpha + \alpha\beta}{\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})} \right] \Delta$$

$$+ \left[\frac{\left((1-\alpha)L_0 + \alpha\beta s_{00} \right) w}{\left((1-\alpha)L_0 + \alpha\beta s_{00} \right) w + 2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10} w}{\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})} \right] d\log w.$$

Plugging this expression for $d\log r$ into (M.12), we get:

$$d\log w \; = \frac{-\gamma \left(\frac{1}{2L_1} + \frac{1}{L_0} \right) + (1-\alpha) \left[\frac{(1-\alpha)w}{((1-\alpha)L_0 + \alpha\beta s_{00})w + 2\alpha\beta s_{01}} + \frac{1}{2} \frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11}+s_{1,-1})} \right]}{\alpha + (1-\alpha) \left[\frac{((1-\alpha)L_0 + \alpha\beta s_{00})w}{((1-\alpha)L_0 + \alpha\beta s_{00})w + 2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10} w}{\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11}+s_{1,-1})} \right]} \Delta.$$

If $\alpha > 1/2$, the denominator in $d\log w$ is always positive. Hence, the stability condition $d\log w > 0$ becomes:

$$\frac{(1-\alpha)\,w}{((1-\alpha)L_0+\alpha\beta s_{00})\,w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}>\frac{\gamma}{1-\alpha}\left(\frac{1}{L_0}+\frac{1}{2L_1}\right). \tag{M.15}$$

When $\alpha > 1/2$, the inequalities (M.14) and (M.15) are necessary and sufficient for an interior equilibrium to be stable.

We now rewrite these two conditions in terms of the variable ρ only. Using Lemma 2 and the equilibrium relationship $\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho)$, as well as $\rho = r^{-\beta\varepsilon}$, $\omega = w^{\varepsilon}$, and $\eta = \alpha\beta/(1-\alpha)$, (M.14) and (M.15) become

$$\frac{f(\rho)+\frac{1}{2}(1+\eta)\rho^{-\frac{1}{\beta\varepsilon}}\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}}{((1+\eta)\rho+2\phi)\,f(\rho)+2\eta\phi\rho}>\frac{\gamma}{1-\alpha}\left(\frac{\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}}{2(\phi\rho+1+\phi^2)}+\frac{1}{\rho+2\phi}\right), \tag{M.16}$$

$$\frac{(1+\eta)\,f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}-\eta\right)\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}}{((1+\eta)\rho+2\phi)\,f(\rho)+2\eta\phi\rho}>\frac{\gamma}{1-\alpha}\left(\frac{\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}}{2(\phi\rho+1+\phi^2)}+\frac{1}{\rho+2\phi}\right), \tag{M.17}$$

Solving the equilibrium condition $f(\rho) = g(\rho;\gamma)$ w.r.t. γ yields

$$\gamma=\frac{\alpha}{1+\varepsilon}\frac{\log\left(\rho^{-\psi}f(\rho)\right)}{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}\right)}.$$

Plugging this expression into (M.16) – (M.17), we get:

$$\begin{aligned}\Phi(\rho)\equiv&\frac{2(1-\alpha)(1+\varepsilon)(\phi\rho+1+\phi^2)}{((1+\eta)\rho+2\phi)\,f(\rho)+2\eta\phi\rho}\\&\times\frac{f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}+\frac{\eta}{2}\rho^{-\frac{1}{\beta\varepsilon}}\right)\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}}{\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}(\rho+2\phi)+2(\phi\rho+1+\phi^2)}\\&\times\frac{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}\right)}{\alpha\log\left(\rho^{-\psi}f(\rho)\right)}>1,\end{aligned}$$

$$\Psi(\rho) \equiv \frac{2(1-\alpha)(1+\varepsilon)(\phi\rho+1+\phi^2)}{((1+\eta)\rho+2\phi)\, f(\rho)+2\eta\phi\rho} \times \frac{(1+\eta)\, f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}-\eta\right)[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}(\rho+2\phi)+2(\phi\rho+1+\phi^2)} \times \frac{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\right)}{\alpha\log\left(\rho^{-\psi}f(\rho)\right)} > 1.$$

Last, we set:

$$\mathbb{F}(\rho) \equiv \min\left\{\Phi(\rho), \Psi(\rho)\right\},$$

which is independent of ρ. Verifying $\mathbb{F}(\rho) > 1$ can be done numerically for any vector of parameters by plotting $\mathbb{F}(\rho)$ as a function of the variable ρ. Q.E.D.

N. Proof of Proposition 6

Under weak increasing returns, given Lemmas 2 and 3, f and g must intersect exactly once. Furthermore, because $f(1) < 1 < g(1;\gamma)$, the intersection must occur at $\rho^* < 1$. Because $\rho^* > \rho_L$, we have

$$f(\rho_L) > f(\rho^*) = g(\rho^*) > g(\rho_L) \tag{N.1}$$

because f is decreasing by Lemma 2 and g is increasing in ρ by Lemma 3. As shown by (D.1), $g(\rho_L)$ is independent of γ. Combining this with (N.1), we obtain $f(\rho_L) - g(\rho_L;\gamma) > 0$. Because $f(\rho^*) - g(\rho^*;\gamma) = 0$ while $f - g$ is decreasing by Lemmas 2 and 3, we have $\rho_L < \rho^*$ for all $\gamma < \alpha/\varepsilon$, which amounts to $\ell^* > 1$.

We now study the impact of γ on (i) ρ^*, (ii) $\omega^*$and (iii) ℓ^*.

(i) Because $\partial g(\rho;\gamma)/\partial\gamma > 0$, applying the implicit function theorem to (25) leads to

$$\frac{d\rho^*}{d\gamma} = \left.\frac{\partial g(\rho;\gamma)/\partial\gamma}{\partial f'(\rho)/\partial\rho - \partial g(\rho;\gamma)/\partial\rho}\right|_{\rho=\rho^*} < 0,$$

where the numerator is positive because $\rho^* > \rho_L$ while the denominator is negative because $f(\rho)$ is decreasing and $g(\rho;\gamma)$ is increasing in ρ.

(ii) Differentiating (23) with respect to γ, we obtain:

$$\frac{1+\varepsilon}{\varepsilon}\omega^{\frac{1}{\varepsilon}}\frac{d\omega^*}{d\gamma} = \frac{df}{d\rho}\frac{d\rho^*}{d\gamma} > 0.$$

(iii) From Lemma 6, $d\ell^*/d\gamma > 0$. Q.E.D.

O. Proof of Proposition 7

Step 1. Consider first the case when the spatial discount factor is small ($\phi < \overline{\phi}$), so that $\rho_m < \rho_L < 1 < \rho_M$ holds. Therefore, for $\Delta > 0$ sufficiently small, we have:

$$\rho_m + \Delta < \rho_L - \Delta < \rho_L + \Delta < 1 < \rho_M.$$

If γ is sufficiently close to α/ε (but still such that $\gamma > \alpha/\varepsilon$ holds), Lemma 3 implies the following inequalities:

$$\begin{aligned} g(\rho_m + \Delta; \gamma) &< f(\rho_m + \Delta), \\ g(\rho_L - \Delta; \gamma) &> f(\rho_L - \Delta), \\ g(\rho_L + \Delta; \gamma) &< f(\rho_L + \Delta), \\ g(\rho_M; \gamma) &> f(\rho_M) = 0, \end{aligned}$$

where the last inequality holds because (24) implies that, for $\gamma > \alpha/\varepsilon$, $g(\rho;\gamma) > 0$ for all $\rho > 0$ while $f(\rho_M) = 0$ for any γ by definition of ρ_M. Therefore, by continuity of f and g, (25) has at least *three* distinct solutions, which we denote as follows:

$$\rho_M > \rho_2^\star > \rho_3^\star.$$

Furthermore, the properties of function g imply the following:

$$\lim_{\gamma\varepsilon \searrow \alpha} \rho_1^\star = \rho_M,$$

$$\lim_{\gamma\varepsilon \searrow \alpha} \rho_2^\star = \rho_L,$$

$$\lim_{\gamma\varepsilon \searrow \alpha} \rho_3^\star = \rho_m.$$

The solution $\rho_2^\star$ matches the equilibrium of Proposition 6. As for the other two solutions, $\rho_1^\star$ and $\rho_3^\star$, when γ is close enough to α/ε, we have $\rho_2^\star > 1 > \rho_3^\star$.

As $\gamma \searrow \alpha/\varepsilon$, it follows from Lemma 3 that $f(\rho_2^\star)$ and $f(\rho_3^\star)$ converge, respectively, to 0 and ∞, which implies:

$$\lim_{\gamma\varepsilon \searrow \alpha} \omega_1^\star = 0 \qquad \text{and} \qquad \lim_{\gamma\varepsilon \searrow \alpha} \omega_3^\star = \infty.$$

Hence, $\omega_1^\star < 1 < \omega_3^\star$ when $\gamma\varepsilon$ is close enough to α. It then follows from (24) that

$$\lim_{\gamma\varepsilon \searrow \alpha} \ell_1^\star = 0 \qquad \text{and} \qquad \lim_{\gamma\varepsilon \searrow \alpha} \ell_3^\star = \infty.$$

Step 2. Consider now the case where the spatial discount factor is high ($\phi > \overline{\phi}$). Then, we know from Lemma 5 that there exists a value $\overline{\eta} \in (0,1)$ such that

$$\rho_L \leq \rho_m < 1 < \rho_M \tag{O.1}$$

is satisfied for $\eta \geq \overline{\eta}$, while $\rho_m < \rho_L < 1 < \rho_M$ holds otherwise. Under (O.1), there is a small $\Delta > 0$ such that the following inequalities hold:

$$g(\rho_M - \Delta; \gamma) < f(\rho_M - \Delta),$$
$$g(\rho_M; \gamma) > f(\rho_M) = 0.$$

while $\rho^\star > 1$ when γ slightly exceeds α/ε.

Furthermore,

$$\lim_{\gamma\varepsilon \searrow \alpha} (\omega_1^*)^{\frac{\varepsilon}{1+\varepsilon}} = f(\rho_M) = 0.$$

Because $\lim_{\gamma\varepsilon \searrow \alpha} \omega_1^\star = 0$, $\omega_1^\star < 1$ when $\gamma\varepsilon$ is sufficiently close to α.

Last, using (7), we have:

$$\lim_{\gamma\varepsilon \searrow \alpha} \ell_1^\star = 0.$$

Q.E.D.

A *P. Proof of Proposition 8*

First, we show the existence and uniqueness of an equilibrium. The equilibrium condition (25) can be restated as follows:

$$\frac{1}{\phi\rho + 1 + \phi^2} \left(\frac{1 + \eta \frac{1+\phi^2 - 2\phi\rho^{1+\frac{1}{\beta\varepsilon}}}{\phi\rho + 1 + \phi^2}}{1 + \eta \frac{\rho - \phi\rho^{-\frac{1}{\beta\varepsilon}}}{\rho + 2\phi}} \right)^\lambda = \rho^\mu \frac{\rho}{\rho + 2\phi}, \tag{P.1}$$

where λ and μ are defined by

$$\lambda \equiv \frac{\gamma\varepsilon - \alpha}{\gamma + \alpha} > 0 \qquad \text{and} \qquad \mu \equiv \frac{\gamma\varepsilon - \alpha - (1-\alpha)(1+\varepsilon)}{\beta\varepsilon(\gamma + \alpha)}.$$

The first term of the LHS of (P.1) decreases in ρ; the second term also decreases because the numerator decreases while the denominator increases in ρ. Therefore, the LHS of (P.1) is a decreasing function of ρ. Furthermore, the RHS of (P.1) increases from 0 to ∞ in ρ when $\mu > 0$. It is readily verified that $\mu > 0$ if and only if

$$\gamma > \frac{1+\varepsilon}{(1-\beta)\varepsilon} - \alpha.$$

Hence, (P.1) has a unique solution ρ^*.

We now show that ℓ^* converges monotonically toward 1 when $\gamma > \gamma_s$ increases. Using (7), we obtain

$$\log \ell^* = -\frac{1}{\gamma\varepsilon/\alpha - 1} \log \left((\rho^*)^{\frac{1}{\eta}} \frac{\rho^* + 2\phi}{\phi\rho^* + 1 + \phi^2} \right). \tag{P.2}$$

Because $\rho^* > \rho_L$ under strong increasing returns, the expression under the log is greater than 1 and thus the RHS of (P.2) is negative. Furthermore, as ρ^* decreases with γ, the RHS of (P.2) increases with γ. In addition, the first of the RHS goes to 0 when γ goes to infinity. Consequently, ℓ^* converges to 1. Q.E.D.