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EQUILIBRIUM LAND USE PATTERNS IN A NONMONOCENTRIC CITY*

Hideaki Ogawa and Masahisa Fujita†

1. INTRODUCTION

In the 1970's, a new genre of mathematical urban land use theory, called the new urban economics (Mills and MacKinnon [8]), was developed. The principal characteristics of those models are the monocentricity of the city, the uniform transportation system, the homogeneity of households and production firms, and so forth (Richardson [16]). Among these, the assumption of monocentricity seems to be very crucial to the formulation of the models. By monocentricity, we assume that the city has a single, pre-specified center, the CBD, which has a fixed size and employs the city's entire labor force. This assumption greatly simplifies the analysis: for example, commuting trips can be exactly specified if the residential locations are known, and with the assumption of a linear or circular city, the spatial characteristics of each location in the city can be described simply by the distance from the CBD.

However, from the viewpoint of theoretical completeness, the centrality or noncentrality of a city should be explained within the framework of the model, which incorporates interdependences among economic activities, without pre-specified locations of employment activities. And, if the model succeeds in clarifying the conditions for the existence of the CBD, we can not only judge the adequacy of the monocentric assumption adopted by the current urban land use models but also step forward to a more fruitful theory which is capable of explaining various kinds of nonmonocentric phenomena. Moreover, from the viewpoint of reality, monocentricity is an implausible assumption. In the face of the tendency of increasing decentralization and a decline of the role of the CBD as a single focus of employment activity (Mills [9]), urban land use models based on the monocentric assumption are clearly inadequate for analyzing recent cities.[1] Thus, both from the viewpoints of theoretical completeness and practical usefulness, the development of nonmonocentric models of urban land use is needed.

There have been a few works which attempt to relax the monocentric assumption in two different ways. One approach is to introduce pre-specified

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†Research Assistant, Department of Architecture, Gifu Technical College, Gifu, Japan and Assistant Professor, Regional Science Department, University of Pennsylvania, respectively.

[1]See Odland [12] for a test for monocentricity.

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multiple centers into the model (Hartwick and Hartwick [6], Lave [7], Odland [12], Papageorgiou and Casetti [15], Romanos [17], White [18]); these may be called *multicentric models.* The other is to construct a more general model without pre-specifying any centers (Amson [2], Beckmann [3], Borukhov and Hochman [4], Capozza [5], Niedercorn [10], Odland [11]); these may be called *nonmonocentric models.* Although the complication and mathematical intractability associated with multicentricity are severe, the multicentric spatial structure can be easily analyzed if we permit specialization of functions among an intraurban hierarchy of centers and introduction of multiple goods. On the other hand, the second approach to relaxing monocentricity has so far no such general scheme in which to formulate the models. But this approach seems to be much more important and fruitful for the future development of urban land use theory.

It is the purpose of this paper to develop a model of nonmonocentric urban land use and to obtain equilibrium land use patterns by taking more explicitly into consideration the nonmonocentric aspects of urban activity.[2] We consider a linear city consisting of households and business firms. The model presented here does not require either employment or residential location to be specified a priori. In this context, the concept of bid rent curves originally defined in Alonso [1] is generalized for the case of the nonmonocentric city. Then, it is shown that the city exhibits three different types of equilibrium land use patterns according to the values of certain parameters. Among the parameters, commuting and transaction costs play a crucial role in determining the explicit patterns of equilibrium land use.

In Section 2, we first state the assumptions of spatial environments and individual behavior, and then formulate a nonmonocentric urban land use model. Next, in Section 3, we generalize the bid rent curves and define them as functions of the wage profile and the distribution of business firms. By using these bid rent curves, the equilibrium conditions are explicitly stated. Then, we observe some basic properties of equilibrium land use patterns. From these properties, only a few land use patterns are left as possibilities. Finally, in Section 4, we obtain equilibrium land use patterns and associated parametric conditions.

2. ASSUMPTIONS AND FORMULATION OF THE MODEL

Consider a long strip of agricultural land of width 1 (unit distance) which is going to be occupied by a city. Assuming that the length of the city will be large compared with its width and that the commuting and transaction costs in the breadthwise direction are negligible, the city is treated as a linear city. Therefore, each location in the city is represented by a point, x, on a line-coordinate whose

[2]Among the nonmonocentric models, Amson [2] has proposed a general analytical framework within a dispersed nonmonocentric urban phenomenon, although he did not give any economic description of individual behavior. Beckmann [3] and Borukhov and Hochman [4] developed one sector nonmonocentric models based on explicit behavioral assumptions. And Capozza [5] developed a model of agglomeration economies by focusing on the interactions between the distribution of employment and population in the urban area. His formulation is similar to ours, but his treatment of distance between firms is less explicit in the context of the interactions between continuously dispersed business firms than the treatment presented in this article.

origin does not necessarily correspond to the CBD, and the whole spatial characteristics of the city are described in this continuous space. The indivisibility of facilities is neglected in this paper.

The city will consist of two economic sectors, households and business firms, which consume land. However, in equilibrium, the city may contain nondeveloped agricultural land within its boundaries. Assuming that the labor and land markers are perfectly competitive everywhere in the city, we next describe the behavior of activity in each sector.

2.1 *Households*

There are N_h identical households which have the same preferences. The utility level, U, of each household depends on the amount of land occupied (lot size), S, and the amount of composite commodity, C, consumed by the household. Thus, the utility function is expressed by

$$U = U(S, C) \tag{1}$$

where $\partial U/\partial S > 0$, $\partial U/\partial C > 0$. The composite commodity is imported from the outside of the city at a unit price. Each household has one worker supplying its labor to a business firm. The income of the household is equal to the wage earned by that worker from that business firm. The only travel in the city is the journey to work, and the commuting cost is proportional to the distance between the residence and the job site. The commuting cost per unit distance, t, is assumed to be a given positive constant. Thus, the budget constraint of a household locating at x and working at x_W is given by

$$W(x_W) = R(x)S + C + td(x, x_W) \tag{2}$$

where $W(x_W)$ is the wage paid by a business firm locating at x_W, $R(x)$ is the land rent for a unit of land at x, and $d(x, x_W) = |x - x_W|$ is the distance between the residence and the job site. The objective of each household is to maximize its utility, (1), subject to the budget constraint, (2), by choosing S, C, x and x_W. It should be noted that, since we exclude the monocentricity assumption, each household must decide its job site, x_W, as well as its residential location, x.

However, for the simplicity of analysis, in this article we consider only the case where the lot size of each household is fixed at some positive constant $\overline{S}$.[3] Accordingly, the objective of the household is equivalent to choosing the residential location, x, and the job site, x_W, so as to maximize the amount of composite commodity

$$\max C = W(x_W) - R(x)\overline{S} - td(x, x_W) \tag{3}$$

Since all the households are assumed to be identical, all the households should achieve the same utility level, U^*, and hence the same consumption of composite commodity, C^*, where $U^* = U(C^*, \overline{S})$.

[3]This assumption may be interpreted either that only one type of housing is technologically possible in the city and the lot size is subject to that technological restriction or that the zoning board of the city regulates the housing lot size to be fixed. We hope to relax this assumption in the future.

2.2 *Business Firms*

There are N_f identical business firms, each producing some kind of service or information which is exported outside the city at a constant price p. The production function of each firm is of an input-output type, whose inputs are land and labor, given by

$$Q = \min\left\{\frac{S}{a_S}, \frac{L}{a_L}\right\} \tag{4}$$

where Q = the amount of output produced, S = land, L = labor, a_S = the land/output ratio, and a_L = the labor/output ratio.

Furthermore, for simplicity, we assume that each business firm produces the same positive constant amount of output, $\overline{Q}$. Consequently, each firm requires $a_S\overline{Q}$ units of land and $a_L\overline{Q}$ units of labor. Thus, assuming that there is no unemployment in the city, we have

$$N_f = \frac{1}{a_L\overline{Q}} N_h \tag{5}$$

Another important assumption for business firms is that production requires transactions (i.e., communications or information exchange) among themselves (Capozza [5], O'Hara [14]).[4]

There are two possible ways to take into account those transactions; one is to incorporate them in the production function, and another is to consider them as an element of the cost or profit function. In our model, the latter approach is adopted. In doing so, we introduce three additional assumptions: (i) each business firm transacts strictly equiprobably (for instance, one per period) with every other business firm in the city, (ii) each transaction is separately performed by some kind of communication method (face-to-face or telephone) and the cost of the transaction between any two business firms is proportional to the distance between them, i.e., $\tau d(x_1, x_2)$, where τ is the transaction cost per unit distance and $d(x_1, x_2)$ is the distance between the two firms locating at x_1 and x_2, and (iii) the transactions produce no external effects such as congestion.

Under these assumptions, a business firm locating at x will transact with every other firm in the city, and its total transaction costs are given by

$$\tau T(x) = \tau \int_{f_1}^{f_2} b(y) d(x, y) dy \tag{6}$$

where $T(x)$ = total transaction distance for a firm locating at x, $b(y) \geq 0$ = density function of business firms at each location y, $d(x, y) = |x - y|$ = distance between two firms locating at x and y, f_1, f_2 = the left and right fringes, respectively, of the

[4] The transactions between business firms are introduced into the model as an element of "agglomeration economies" which are often pointed out to be a major characteristic of urban production and one of the main reasons for the existence of cities. These are external economies that result from an advantageous location providing access to other firms. We assume that all the business firms in the city are identical from the viewpoint of location behavior, but they may be different in some other aspects, for example, contents of service or information produced by them.

city, that is,

$$(7) \qquad \begin{cases} f_1 = \min\{x \mid b(x) > 0 \text{ or } h(x) > 0\} \\ f_2 = \max\{x \mid b(x) > 0 \text{ or } h(x) > 0\} \end{cases}$$

and $h(x) \geq 0$ = density function of households at each location x. The slope and curvature of the total transaction distance function are, respectively, given by

$$(8) \qquad \begin{aligned} \frac{dT(x)}{dx} &= \frac{d}{dx}\int_{f_1}^{f_2} b(y)\,|x - y|\,dy \\ &= \frac{d}{dx}\int_{f_1}^{x} b(y)(x - y)dy + \frac{d}{dx}\int_{x}^{f_2} b(y)(y - x)dy \\ &= \int_{f_1}^{x} b(y)dy - \int_{x}^{f_2} b(y)dy \end{aligned}$$

$$(9) \qquad \frac{d^2T(x)}{dx^2} = 2b(x) \geq 0$$

Observe that the first term of the last equality in (8) is the number of business firms to the left of x and the second term is the number of business firms to the right of x. Accordingly, the slope of the total transaction distance function is zero at some x where half the business firms are on each side. And from (9), $T(x)$ is strictly concave at x where $b(x) > 0$, and linear if $b(x) = 0$.

From (4) and (6), profits of a firm locating at x are given by

$$(10) \qquad \pi = p\overline{Q} - R(x)a_S\overline{Q} - W(x)a_L\overline{Q} - \tau T(x)$$

Thus, the objective of the business firm is essentially to choose its location, x, considering the distribution of all other business firms in the city, so as to maximize its profits given by (10). Since all the business firms are assumed to be identical, the profit level of all firms must be the same at the equilibrium regardless of their locations.

3. EQUILIBRIUM CONDITIONS AND SOME BASIC PROPERTIES

3.1 *Equilibrium Conditions and Bid Land Rents*

Having described the behavior of the activity unit in each sector, the rest of our task is to obtain the equilibrium solution for the following set of unknown functions and variables: (a) household density function—$h(x)$, (b) business firm density function—$b(x)$, (c) land rent profile—$R(x)$, (d) wage profile—$W(x)$, (e) commuting pattern—$P(x, x_W)$, and (f) utility level—U^*—and profit level—π^*, for $f_1 \leq x \leq f_2$, where f_1 and f_2 are urban fringe distances defined by (7), and

$$(11) \qquad P(x, x_w) = \frac{\text{number of households locating at } x \text{ and commuting to job site } x_w}{\text{total number } (h(x)) \text{ of households locating at } x}$$

Then, the necessary and sufficient conditions for these variables, $h(x)$, $b(x)$, $R(x)$,

$W(x)$, $P(x, x_W)$, U^*, and π^* to represent an equilibrium land use pattern are summarized as follows.

3.1.1 *Land Market*

(12) $$R(x) = \max\{\Psi(x), \phi(x), R_A\} \qquad \text{at each } x \in [f_1, f_2]$$

(13) $$R(x) = \Psi(x) \text{ if } h(x) > 0 \qquad \text{at each } x \in [f_1, f_2]$$

(14) $$R(x) = \Phi(x) \text{ if } b(x) > 0 \qquad \text{at each } x \in [f_1, f_2]$$

(15) $$R(f_1) = R(f_2) = R_A$$

(16) $$a_S\overline{Q}b(x) + \overline{S}h(x) + (\text{land for agricultural use}) = 1 \qquad \text{at each } x \in [f_1, f_2]$$

3.1.2 *Labor Market*

(17) $$a_L\overline{Q}b(x) = \int_{f_1}^{f_2} h(y)P(y, x)dy \qquad \text{at each } x \in [f_1, f_2]$$

(18) $$\Psi(x) \equiv \Psi(x \mid \underset{\sim}{W}(x_W), U^*) = \max_{x_W}\left\{\frac{1}{\overline{S}}(W(x_W) - C^* - td(x, x_W)) \mid U(\overline{S}, C^*) = U^*\right\}$$

(19) $$\Phi(x) \equiv \Phi(x \mid W(x), \underset{\sim}{b}(x), \pi^*) = \frac{1}{a_s\overline{Q}}(p\overline{Q} - \pi^* - W(x)a_L\overline{Q} - \tau T(x))$$

where $\underset{\sim}{W}(x)$ = wage profile over all x, $W(x)$ = the value of $\underset{\sim}{W}(x)$ at x, $\underset{\sim}{b}(x)$ = the distribution of business firms over all x, and R_A = agricultural land rent (exogenously given).

In the above, functions $\Psi(x)$ and $\Phi(x)$ represent bid land rents of household and business firm, respectively, in the context of nonmonocentric model. Although the bid land rents originally defined in Alonso [1] are solely functions of the distance from the predetermined center, in this article, they are functions of the wage profile and the distribution of business firms as well as the distance from the arbitrarily chosen origin, as defined in (18) and (19). Thus, function $\Psi(x \mid \underset{\sim}{W}(x_W), U^*)$ should be read as the bid land rent of a household locating at x, given the wage profile $\underset{\sim}{W}(x_W)$, corresponding to a utility level U^*; and $\Phi(x \mid W(x), \underset{\sim}{b}(x), \pi^*)$ as the bid land rent of a business firm locating at x, given wage $W(x)$ and the distribution of business firms $\underset{\sim}{b}(x)$, corresponding to a profit level π^*. Note that, in the definition of (18), each household locating at x should have optimally chosen its job site, x_W, considering the trade-off between commuting costs $td(x, x_W)$ and wage $W(x_W)$.

With these definitions in mind, condition (12) says that each piece of land must be occupied by a household, a business firm, or a farm which bids the highest rent at each location. Conditions (13) and (14) claim that if households or business firms locate at x respectively, then they must have succeeded in bidding for land at that location. The boundary condition and the physical constraint on the amount of land are given by (15) and (16), respectively. With respect to the labor market, the demand for labor must be equal to the supply of labor at all locations in the city, as given by (17). Together with (5), condition (17) assures no unemployment in the city. And, in the above, equilibrium profit π^* is implicitly assumed to be nonnegative.

3.2 *Some Terminology*

Let us introduce the following terms:

(i) *(Exclusive) Residential Area:* an area in which only households locate, i.e., $h(x) > 0$ and $b(x) = 0$,

(ii) *(Exclusive) Business District:* an area in which only business firms locate, i.e., $h(x) = 0$ and $b(x) > 0$,

(iii) *Integrated District:* an area in which both households and business firms locate, i.e., $h(x) > 0$ and $b(x) > 0$, and

(iv) *Vacant Land:* agricultural land which is surrounded by residential areas or business districts or integrated districts, i.e., $h(x) = 0$, $b(x) = 0$ and $x \in (f_1, f_2)$.

In the following, a residential area and a business district mean an exclusive residential area and an exclusive business district, respectively.

We classify all the possible land use patterns into the following three mutually exclusive categories:

(A) *Land Use Patterns with Vacant Land:* any land use pattern with vacant land in the city, i.e., $h(x) = 0$ and $b(x) = 0$ at some $x \in (f_1, f_2)$,

(B) *Connecting Land Use Patterns:* any land use pattern consisting of residential areas and business districts with no vacant land, i.e.,
$h(x) > 0$ and $b(x) = 0$, or $h(x) = 0$ and $b(x) > 0$ for all $x \in [f_1, f_2]$,

(C) *Mixed Land Use Patterns:* any land use pattern including integrated districts with no vacant land,

(C-1) *Completely Mixed Patterns:* $h(x) > 0$ and $b(x) > 0$ for all $x \in [f_1, f_2]$,

(C-2) *Incompletely Mixed Patterns:* $h(x) > 0$ and $b(x) > 0$ at some $x \in [f_1, f_2]$, $h(x') > 0$ and $b(x') = 0$, or $h(x') = 0$ and $b(x') > 0$ for all other $x' \in [f_1, f_2]$.

3.3 *Some Basic Properties of Equilibrium Land Use Patterns*

To obtain equilibrium land use patterns, we examine some basic properties of them in this section.

3.3.1 *Commuting Pattern and Wage Profile*

In contrast to the single-destination (the CBD) commuting pattern in monocentric models, the multiplicity of destinations (alternative job sites) considered by each household imposes a difficult task for identifying the commuting pattern. We here explore a fundamental property of the equilibrium commuting pattern.

First, consider two arbitrarily chosen households which are, respectively, locating at a and commuting to a business firm at x and locating at b and commuting to a business firm at y. In this case, if $a < b$ and $y < x$, then we call their journeys to work a *cross commuting* pattern. Second, consider a household locating at a and two alternative job sites x and y. At the equilibrium, the disposable income for land and composite commodity should be equal to the wage less the commuting cost. Then, if $W(x) - td(a, x) > W(y) - td(a, y)$, we say that the household has a *strict preference for the job site* x (compared with y) since the disposable income associated with job site x is higher than that associated with y. Third, take two households and their job sites, respectively. If each of them has a strict preference for its job site (compared with the other household's job site) and their commuting pattern is one of cross commuting, then we call this a pattern of *cross commuting with strict preferences on job sites.* In the context of these terminologies, we have:

Property 1: In any equilibrium land use pattern, there is no cross commuting with strict preferences on job sites.

Proof: There are six different cases for cross commutings: $y < x \le a < b$, $y < a < x \le b$, $y \le a < b < x$, $a \le y < x \le b$, $a < y < b \le x$, and $a < b \le y < x$, where a and x are, respectively, the residential location and the job site of a household, and b and y are those of another household. In any case, in order for a household locating at a to strictly prefer commuting to x to commuting to y, the following relation must hold under any equilibrium wage profile:

$$W(x) - t|a - x| > W(y) - t|a - y| \tag{20}$$

Similarly, for a household locating at b and commuting to y,

$$W(x) - t|b - x| < W(y) - t|b - y| \tag{21}$$

From (20) and (21), we have

$$|a - y| + |b - x| > |a - x| + |b - y| \tag{22}$$

(Case 1): $y < x \le a < b$. Equation (22) should be rewritten as

$$a - y + b - x > a - x + b - y \tag{22a}$$

Obviously, the left-hand side of (22a) is the same as the right-hand side, and therefore, (22) cannot be true.

(Case 2): $y < a < x \le b$. Equation (22) should be rewritten

$$a - y + b - x > x - a + b - y \tag{22b}$$

and thus $a > x$, which contradicts the assumption that $a < x$. Since every other case can be eliminated by similar arguments, we conclude that there should be no cross commuting with strict preferences on job sites at the equilibrium.

Property 1 holds if both households exhibit a strict preference for job sites. We should next examine the possibility of cross commuting under different situations. It can be verified that cross commuting is possible only when neither household has a strict preference for its job site and only in two cases: $y < x \le a < b$ or $a < b \le y < x$ where a and x are, respectively, the residential location and the job site of one household, and b and y are those of another household. As for all the other cases, it can be demonstrated that there is no cross commuting. Therefore, whenever there is a cross commuting, neither household has a strict preference for its job site (compared with the others); hence, by exchanging the job sites of these two households, we can obtain an equilibrium commuting pattern without cross commuting. Thus, every possible commuting pattern at the equilibrium can be equivalently considered as a no cross commuting type.

Next, the wage profile associated with an equilibrium land use pattern with commuting will be examined. Suppose that commuting takes place in the city, and consider any two households arbitrarily chosen: one locating at a and commuting to x, another locating at b and commuting to y. Then, by using Property 1, we obtain the following relationship in the corresponding wage profile.

Property 2: Consider any two households arbitrarily chosen; one locating at a and commuting to x, another locating at b and commuting to y. Then, (1) if $a < b \leq x < y$, $dW(z)/dz = t$ at each $z \epsilon (x, y)$ where $b(z) > 0$, (2) if $x < y \leq a < b$, $dW(z)/dz = -t$ at each $z \epsilon (x, y)$ where $b(z) > 0$. Namely, if commuting takes place in the equilibrium city, then the equilibrium wage profile must be a linear function of distance on the corresponding business district.

Proof: First, consider the case of $a < b \leq x < y$ (Figure 1). Take a firm at x' where $x' \epsilon (x, y)$ and $b(x') > 0$. Then, from Property 1, a household must exist at some $a' \epsilon [a, b]$ which is commuting to that firm. In this event, for a household locating at a, its disposable income for land and composite commodity when working at x must be at least as much as when working at x'. Therefore,

$$W(x) - t(x - a) \geq W(x') - t(x' - a) \tag{23}$$

Similarly, for a household locating at a' and working at x',

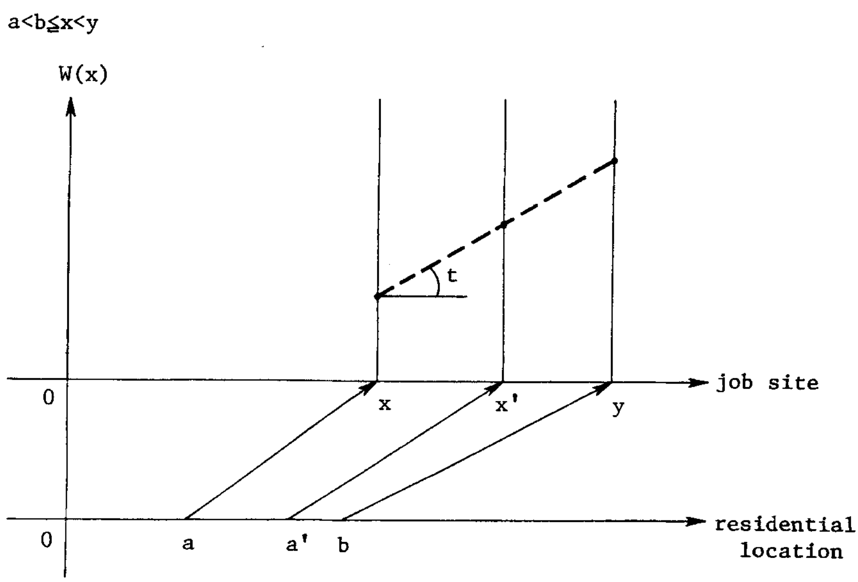

FIGURE 1: Wage Profile Associated With the Journey to Work.

(24) $$W(x') - t(x' - a') \geq W(x) - t(x - a')$$

From (23), we get $W(x') - W(x) \leq t(x' - x)$, and from (24), $W(x') - W(x) \geq t(x' - x)$, and thus

(25) $$W(x') - W(x) = t(x' - x)$$

Since x' is an arbitrary point between x and y such that $b(x') > 0$, from (25) we conclude that $dW(z)/dz = t$ at each $z \in (x, y)$ where $b(z) > 0$. The case of $x < y \leq a < b$ can be proved in the same way.

On the contrary, the following property holds in the integrated district.

Property 3: At the equilibrium, $x = x_W$ for all households in an integrated district. Namely, each household in an integrated district must choose its job site at its residential location.

The above property can be proved by showing that if $x \neq x_W$ for some household in the integrated district, then $\Psi(x)$ is not equal to $\Phi(x)$ in the region where commuting takes place, and hence, the equilibrium conditions for the land market are violated.[5]

3.3.2 *Vacant Land*

We next observe

Property 4: There is no vacant land in any equilibrium land use pattern.

This property can be proved by observing that there always exist incentives of relocation into the vacant land in question for some households because of savings in commuting costs and land rent.[6] In consequence of this property, we immediately claim that any land use pattern with vacant land cannot be in equilibrium.

3.3.3 *Business District*

As for business districts in the equilibrium land use pattern, we observe the following properties.[7]

Property 5: In any equilibrium land use pattern, a business district cannot locate at either fringe of the city.

Property 6: A connecting land use pattern with more than one business district cannot be in equilibrium.

From these two properties, only one business district may exist at the center of a connecting land use pattern if it is in equilibrium.

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[5]For a detailed proof of Property 3, see Ogawa and Fujita [13].

[6]It is not difficult to see this property intuitively. The detailed proof of Property 4 is given in Ogawa and Fujita [13].

[7]For proofs of Properties 5 and 6, see Ogawa and Fujita [13].

4. EQUILIBRIUM LAND USE PATTERNS

From properties in Section 3.3, there remain only a few land use patterns as possibilities; one in each of three categories (B), (C-1), and (C-2).[8] In this section, we will show that these three land use patterns are equilibrium ones if the parameters satisfy certain conditions.

4.1 *Connecting Land Use Pattern with CBD*

In the category of connecting land use pattern, from Properties 5 and 6, only one pattern consisting of one business district at the center and two residential areas is possible.

By choosing the origin at the center of the business district, the land use pattern of the city can be depicted as in Figure 2 (A). From the following conditions

$$\int_{-e}^{e} a_L\overline{Q}b(x)dx = N_h,\ b(x) = \frac{1}{a_S\overline{Q}} \qquad \text{for } x \in [-e, e] \tag{26}$$

$$\int_{e}^{f} h(x)dx = \frac{1}{2}N_h,\ h(x) = \frac{1}{\overline{S}} \qquad \text{for } x \in [e, f] \tag{27}$$

boundary distance e and fringe distance f are obtained

$$e = \frac{a_S}{2a_L}N_h \qquad f = \frac{a_S + a_L\overline{S}}{2a_L}N_h$$

By the symmetry of the land use pattern,[9] it suffices to examine the right half of the city. From Property 2,

$$W(x) = W_0 - tx \tag{28}$$

where W_0 is the wage paid by the business firms at the origin, which is yet unknown.[10] And, from (19),

$$\Phi(x) = \frac{1}{a_S\overline{Q}}(p\overline{Q} - \pi^* - W(x)a_L\overline{Q} - \tau T(x)) \tag{29}$$

Thus, using (8) and (9), we have

$$\Phi'(x) = -\frac{\tau}{a_S\overline{Q}}T'(x) - \frac{a_L}{a_S}W'(x)\begin{cases} > 0 \text{ for } x \in \left[0, \frac{a_La_S\overline{Q}^2}{2}t\right) \\ = 0 \text{ at } x = \frac{a_La_S\overline{Q}^2}{2}t \\ < 0 \text{ for } x \in \left(\frac{a_La_S\overline{Q}^2}{2}t, f\right] \end{cases} \tag{30}$$

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[8]There exist some other land use patterns which do not violate any properties. However, it can be easily shown that they violate some equilibrium conditions.

[9]The proof of symmetry in this land use pattern can be found in Ogawa and Fujita [13].

[10]The equilibrium wage profile, $W(x)$, in the residential areas can be under the dotted line in Figure 2(A) as long as that wage curve keeps the condition $R(x) = \Psi(x) \geq \Phi(x)$ at each $x \in [-f, -e]$ and $x \in [e, f]$. But we can show that this does not change our conclusion.

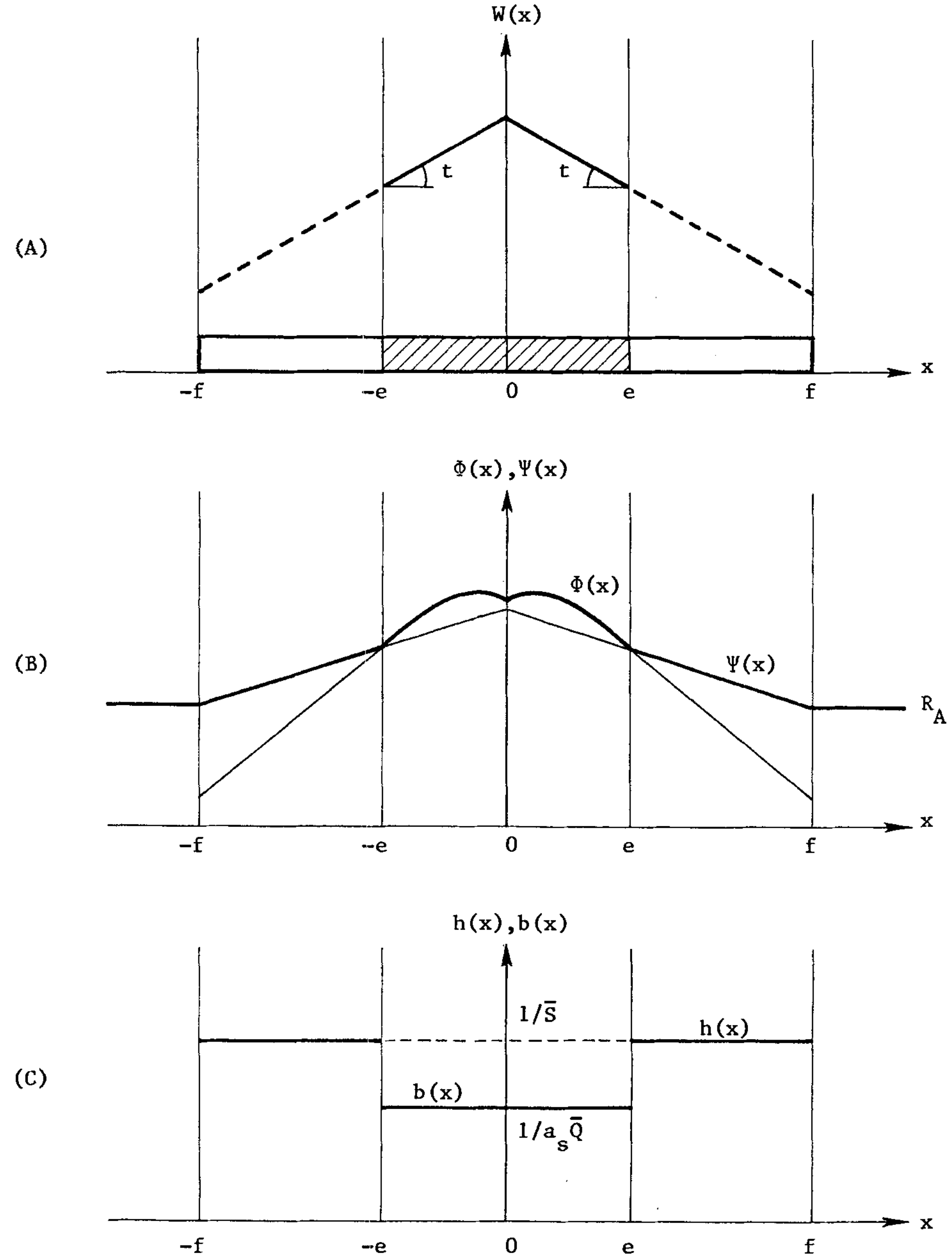

FIGURE 2: An Equilibrium Land Use Pattern: Connecting Land Use with CBD for $0 < t \leq [\tau\overline{S}N_h/2a_L\overline{Q}^2(a_S + a_L\overline{S})]$.

(31) $$\Phi''(x) = -\frac{\tau}{a_S\overline{Q}}T''(x)\begin{cases} < 0 \text{ for } x \in [0, e] \\ = 0 \text{ for } x \in (e, f] \end{cases}$$

From (30) and (31), the bid rent function $\Phi(x)$ can be depicted as in Figure 2(B). On the other hand, from (18) and (28), $\Psi(x)$ is obtained as follows

(32) $$\Psi(x) = \frac{1}{\overline{S}}(W_0 - C^*) - \frac{t}{\overline{S}}x$$

Thus, the bid rent function $\Psi(x)$ can be depicted as in Figure 2(B). These bid rent functions must satisfy the following equilibrium conditions:

(33) $$R(x) = \Phi(x) \geq \Psi(x) \qquad \text{for } x \in [0, e)$$

(34) $$R(x) = \Phi(x) = \Psi(x) \qquad \text{at } x = e$$

(35) $$R(x) = \Psi(x) \geq \Phi(x) \qquad \text{for } x \in (e, f]$$

(36) $$R(x) = \Psi(x) = R_A \qquad \text{at } x = f$$

If (34) is satisfied, then condition (35) can be replaced by $\Phi'(e) \leq \Psi'(e)$. That is,

$$\begin{aligned}\Phi'(e) - \Psi'(e) &= -\frac{\tau}{a_S\overline{Q}}T'(e) + \frac{a_L}{a_S}t + \frac{t}{\overline{S}} \\ &= -\frac{\tau N_h}{a_S a_L \overline{Q}^2} + \frac{a_S + a_L\overline{S}}{a_S\overline{S}}t \leq 0\end{aligned}$$

But, if $\Phi'(e) = \Psi'(e)$, condition (33) cannot be satisfied. Hence, for the connecting land use pattern in Figure 2(A) to be equilibrium, the following condition must be satisfied:

(37) $$t < \frac{\tau\overline{S}N_h}{a_L\overline{Q}^2(a_S + a_L\overline{S})}$$

Next, $\Phi(x)$ is strictly concave in the area [0, e]. Thus, if (34) is satisfied, condition (33) can be replaced by $\Phi(0) \geq \Psi(0)$. Thus, using (29) and (32), we have

(38) $$\frac{a_S + a_L\overline{S}}{a_S\overline{S}}W_0 \leq \frac{1}{a_S\overline{Q}}(p\overline{Q} - \pi^*) - \frac{\tau N_h^2}{4a_L^2\overline{Q}^2} + \frac{C^*}{\overline{S}}$$

And from (29), (32) and (34), we have

(39) $$\frac{a_S + a_L\overline{S}}{a_S\overline{S}}W_0 = \frac{a_S + a_L\overline{S}}{2a_L\overline{S}}N_h t + \frac{1}{a_S\overline{Q}}(p\overline{Q} - \pi^*) - \frac{\tau N_h^2}{2a_L^2\overline{Q}^2} + \frac{C^*}{\overline{S}}$$

From (38) and (39), we get

(40) $$t \leq \frac{\tau\overline{S}N_h}{2a_L\overline{Q}^2(a_S + a_L\overline{S})}$$

Combining (37) and (40), we can conclude that the land use pattern depicted in Figure 2(A) is equilibrium if and only if conditions (36) and (40) hold. The corresponding equilibrium land rent $R(x)$ is obtained from (29), (32) and (36) as follows.

$$R(x) = \begin{cases} \left(\frac{a_L}{a_s}|x| + \frac{a_s - a_L\overline{S}}{2a_L\overline{S}} N_h\right)t + \frac{\tau}{a_s\overline{Q}}\left(\frac{a_s N_h^2}{2a_L^2\overline{Q}^2} - T(x)\right) + R_A & \text{for } x \in [-e, e] \\ -\frac{t}{\overline{S}}|x| + \frac{t}{\overline{S}}\frac{a_s - a_L\overline{S}}{2a_L} N_h + R_A & \text{for } x \in [-f, -e], x \in [e, f] \\ R_A & \text{for } x \in [-\infty, -f], x\, [f, \infty] \end{cases}$$

For $x \in [0, e]$, $R(x) = \Phi(x)$ and $T(x)$ increases at an increasing rate while $W(x)$ decreases linearly, and consequently, $R(x)$ increases as the distance from the origin increases, attains its maximum at $x = (1/2)a_L a_s \overline{Q}^2 t$, and then falls.

However, the equilibrium utility level $U^* = U(\overline{S}, C^*)$, profit level π^*, and wage profile (particularly W_0) cannot be uniquely determined within the model, which is due to the simple input-output production function. If one of those three variables is exogenously specified, then the other two variables can be uniquely determined.

4.2 *Completely Mixed Land Use Pattern*

Take the origin at the center of the city (see Figure 3). From Property 3, the density functions of households and business firms are given by

$$h(x) = \frac{a_L}{a_s + a_L\overline{S}} \qquad b(x) = \frac{1}{\overline{Q}(a_s + a_L\overline{S})}$$

By solving $\int_{-f}^{f} h(x)dx = N_h$ for f, we get the fringe of the city $f = [(a_s + a_L\overline{S})/2a_L]N_h$.

Next, the equilibrium conditions in the land market are, for $x \in [-f, f]$

(41) $$R(x) = \Psi(x) = \Phi(x)$$

(42) $$\Psi(x) = \frac{1}{\overline{S}}(W(x) - C^*)$$

(43) $$\Phi(x) = \frac{1}{a_S\overline{Q}}(p\overline{Q} - \pi^* - \tau T(x)) - \frac{a_L}{a_S}W(x)$$

(44) $$R(x) = \Psi(x) = \Phi(x) = R_A \qquad \text{at } x = -f, f$$

From (41), (42) and (43), we have

$$\frac{1}{\overline{S}}(W(x) - C^*) = \frac{1}{a_S\overline{Q}}(p\overline{Q} - \pi^* - \tau T(x)) - \frac{a_L}{a_S}W(x)$$

Solving the above equality for $W(x)$, we get

(45) $$W(x) = \frac{p\overline{Q} - \pi^* - \tau T(x)}{\overline{Q}(a_S + a_L\overline{S})}\overline{S} + \frac{a_S C^*}{a_S + a_L\overline{S}}$$

Since $W'(x) = -\tau\overline{S}T'(x)/\overline{Q}(a_S + a_L\overline{S})$ and $W''(x) = -\tau\overline{S}T''(x)/\overline{Q}(aS + a_L\overline{S}) < 0$,

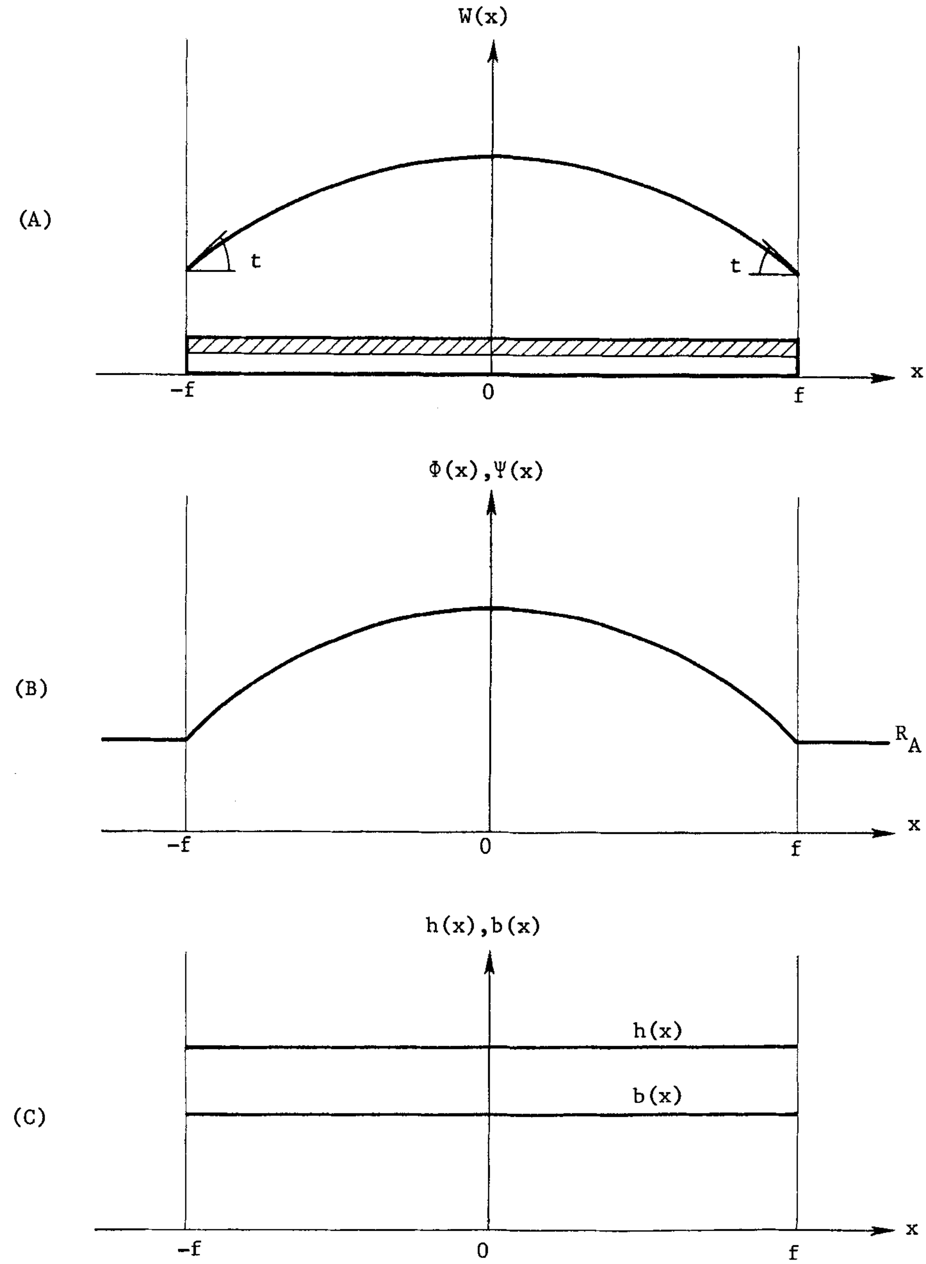

FIGURE 3: An Equilibrium Land Use Pattern: Completely Mixed Land Use for $t \geq [\tau \overline{S} N_h / a_L \overline{Q}^2 (a_S + a_L \overline{S})]$.

$W(x)$ is a strictly concave function and has its maximum at $x = 0$. Substituting (45) into (43), bid rent function $\Phi(x)$ is obtained

$$\Phi(x) = \frac{p\overline{Q} - \pi^* - \tau T(x)}{\overline{Q}(a_S + a_L\overline{S})} - \frac{a_L C^*}{a_S + a_L\overline{S}} \tag{46}$$

Again, since $\Phi'(x) = -\tau T'(x)/\overline{Q}(a_S + a_L\overline{S})$ and $\Phi'(x) = -\tau T''(x)/\overline{Q}(a_S + a_L\overline{S}) < 0$, $\Phi(x)$ $(= \Psi(x))$ is also a strictly concave function.

Finally, conditions (18) and (42) imply that $|W'(x)| \leq t$ for all $x \in [-f, f]$, and since $W(x)$ is a strictly concave function, we have

$$W'(f) \geq -t \quad \text{and} \quad W'(-f) \leq t \tag{47}$$

which holds, from (45), if and only if

$$t \geq \frac{\tau\overline{S}N_h}{a_L\overline{Q}^2(a_S + a_L\overline{S})} \tag{48}$$

Accordingly, the completely mixed land use with no commuting is the equilibrium pattern if and only if (44) and (48) are satisfied. Equilibrium land rent $R(x)$ is given by

$$R(x) = \begin{cases} -\dfrac{\tau}{\overline{Q}(a_S + a_L\overline{S})} T(x) + \dfrac{\tau N_h^2}{2a_L^2\overline{Q}^2} + R_A & \text{for } x \in [-f, f] \\ R_A & \text{for } x \in [-\infty, -f], x \in [f, \infty] \end{cases}$$

Again, equilibrium utility level U^*, profit level π^*, and wage profile $W(x)$ cannot be uniquely determined within the model.

4.3 *Incompletely Mixed Land Use Pattern*

First, the incompletely mixed land use pattern must be symmetric if it is an equilibrium pattern.[11] Next, we will show that the symmetric incompletely mixed land use pattern depicted in Figure 4(A) is really equilibrium under certain conditions on parameters. We continue to consider the right half of the city.

From Property 3, density functions are given by

$$h(x) = \begin{cases} \dfrac{a_L}{a_S + a_L\overline{S}} & \text{for } x \in [0, e] \\ \dfrac{1}{\overline{S}} & \text{for } x \in [f, g] \end{cases}$$

$$b(x) = \begin{cases} \dfrac{1}{\overline{Q}(a_S + a_L\overline{S})} & \text{for } x \in [0, e] \\ \dfrac{1}{a_S\overline{Q}} & \text{for } x \in [e, f] \end{cases}$$

[11]The proof of symmetry in the land use pattern can be found in Ogawa and Fujita [13].

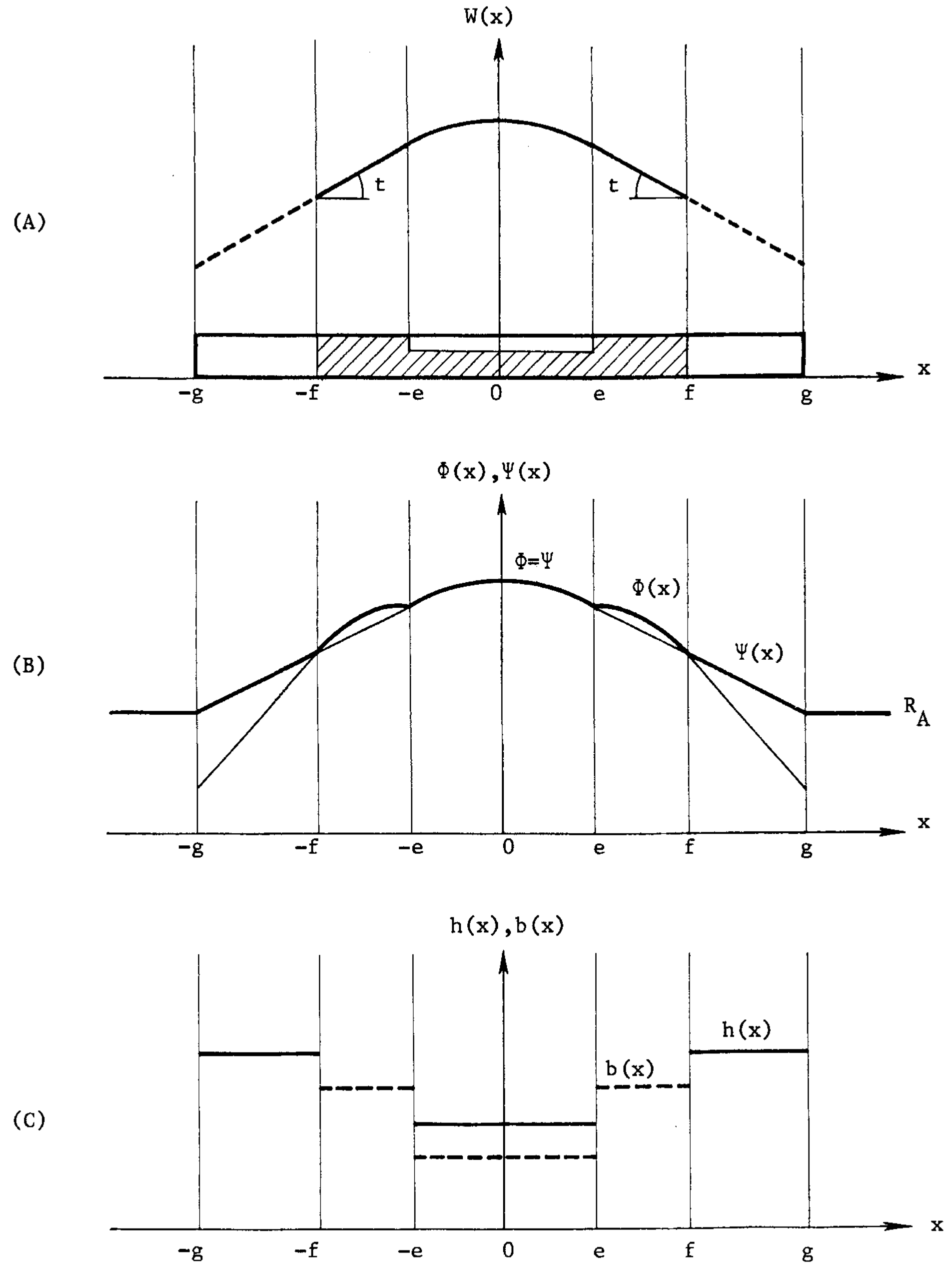

FIGURE 4: An Equilibrium Land Use Pattern: Incompletely Mixed Land Use for $[\tau\overline{S}N_h/2a_L\overline{Q}^2(a_S + a_L\overline{S})] < t < [\tau\overline{S}N_h/a_L\overline{Q}^2(a_S + a_L\overline{S})]$.

The boundary, f, between the business district and the residential area, and the fringe, g, of the city are given by

$$f = \frac{a_L\bar{S}}{a_S + a_L\bar{S}} e + \frac{a_s}{2a_L} N_h \qquad g = \frac{a_S + a_L\bar{S}}{2a_L} N_h$$

In the integrated district, from Property 3, all the households must work and live in the same location. All the households in the residential area commute to business firms in the business district. Thus, from Property 2, the wage profile in the business district is a linear function of distance with slope $-t$. And, the bid rent functions $\Psi(x)$ and $\Phi(x)$, and the wage profile, $W(x)$, must satisfy the following conditions.

$$R(x) = \Psi(x) = \Phi(x) \qquad \text{for } x \in [0, e] \tag{49}$$

$$R(x) = \Phi(x) \geq \Psi(x) \qquad \text{for } x \in (e, f) \tag{50}$$

$$R(x) = \Phi(x) = \Psi(x) \qquad \text{at } x = f \tag{51}$$

$$R(x) = \Psi(x) \geq \Phi(x) \qquad \text{for } x = (f, g] \tag{52}$$

$$R(x) = \Psi(x) = R_A \qquad \text{at } x = g \tag{53}$$

$$\Psi(x) = \frac{1}{\bar{S}}(W(x) - C^*) \tag{54}$$

$$\Phi(x) = \frac{1}{a_S\bar{Q}}(p\bar{Q} - \pi^* - \tau T(x)) - \frac{a_L}{a_S} W(x) \tag{55}$$

From (49), (54) and (55), we have the equilibrium wage profile and bid rent functions in the integrated district as follows.

$$W(x) = \frac{p\bar{Q} - \pi^* - \tau T(x)}{\bar{Q}(a_S + a_L\bar{S})}\bar{S} + \frac{a_S C^*}{a_S + a_L\bar{S}} \tag{56}$$

$$\Phi(x) = \Psi(x) = \frac{p\bar{Q} - \pi^* - \tau T(x)}{\bar{Q}(a_S + a_L\bar{S})} - \frac{a_L C^*}{a_S + a_L\bar{S}} \tag{57}$$

Functions $W(x)$ and $\Phi(x)$ $(= \Psi(x))$ are strictly concave in the integrated district. In the business district and the residential area

$$\Psi''(x) = \frac{1}{\bar{S}} W''(x) = 0 \qquad \text{for } x \in [e, g]$$

$$\Phi''(x) = -\frac{\tau}{a_S\bar{Q}} T''(x) - \frac{a_L}{a_S} W''(x) \begin{cases} < 0 \text{ for } x \in [e, f] \\ = 0 \text{ for } x \in (f, g] \end{cases}$$

From this observation, we see that condition (50) is automatically satisfied if (49) and (51) hold; and condition (52) can be replaced by the condition, $\Psi'(x) >$

$\Phi'(x)$ for $x \in [f, g]$, namely by the following condition.

$$\Psi'(x) - \Phi'(x) = \frac{1}{\overline{S}} W'(x) + \frac{\tau}{a_S \overline{Q}} T'(x) + \frac{a_L}{a_S} W'(x)$$
$$= -\frac{1}{\overline{S}} t + \frac{\tau}{a_S \overline{Q}} \frac{N_h}{a_L \overline{Q}} - \frac{a_L}{a_S} t > 0$$

which implies

$$(58) \qquad t < \frac{\tau \overline{S} N_h}{a_L \overline{Q}^2 (a_S + a_L \overline{S})}$$

Next, from (49) and (51), we have

$$(59) \qquad \frac{a_S + a_L \overline{S}}{a_S \overline{S}} W(e) = \frac{1}{a_S \overline{Q}} (p\overline{Q} - \pi^* - \tau T(e)) + \frac{C^*}{\overline{S}}$$

$$(60) \qquad \frac{a_S + a_L \overline{S}}{a_S \overline{S}} W(e) = \frac{1}{a_S \overline{Q}} (p\overline{Q} - \pi^* - \tau T(f)) + \frac{C^*}{\overline{S}} - \frac{t}{\overline{S}} e + \frac{a_S + a_L \overline{S}}{2 a_L \overline{S}} t N_h$$

which imply

$$(61) \qquad t = \frac{\tau}{a_S \overline{Q}} \left(\frac{a_S N_h^2}{4 a_L^2 \overline{Q}} - \frac{a_S e^2}{\overline{Q} (a_S + a_L \overline{S})^2} \right) \Big/ \left(\frac{a_S + a_L \overline{S}}{2 a_L \overline{S}} N_h - \frac{e}{\overline{S}} \right)$$

Then, when e changes from 0 to $[(a_S + a_L \overline{S}/2a_L]\ N_h$, t increases from $\tau \overline{S} N_h / [2 a_L \overline{Q}^2 (a_S + a_L \overline{S})]$ to $\tau \overline{S} N_h / [a_L \overline{Q}^2 (a_S + a_L \overline{S})]$ monotonically. If t satisfies (58) and (61), it can be easily verified that there is no incentive for any household in the residential areas to change its commuting destination. As for all households in the integrated district, there is also no such incentive if $W(x)$, given by (56), satisfies the condition $W'(e) \geq -t$, which means

$$(62) \qquad t \geq \frac{2 \tau \overline{S}}{\overline{Q}^2 (a_S + a_L \overline{S})^2} e$$

But if (61) is satisfied, then (58) and (62) are automatically satisfied, whatever values e takes between 0 and $[(a_S + a_L \overline{S})/2a_L]\ N_h$. Therefore, equilibrium conditions (49) to (52) are satisfied if and only if (61) holds.

Accordingly, if conditions (53) and (61) are satisfied, the incompletely mixed land use is an equilibrium pattern. The corresponding wage profile, bid rent functions, and density functions are summarized in Figure 4. As before, the wage profile, equilibrium utility level U^*, and profit level π^* cannot be uniquely determined. Equilibrium land rent $R(x)$, however, can be uniquely obtained as a function of R_A and the boundary distance e from the boundary conditions.

The results of the analysis in this section are summarized in Figure 5. From this figure we see that when the commuting rate t is relatively small and/or the transaction rate τ is considerably high, the equilibrium city is characterized by a connecting land use pattern and it has a single business district at the center. On the contrary, when t is considerably high and/or τ is fairly small, the completely

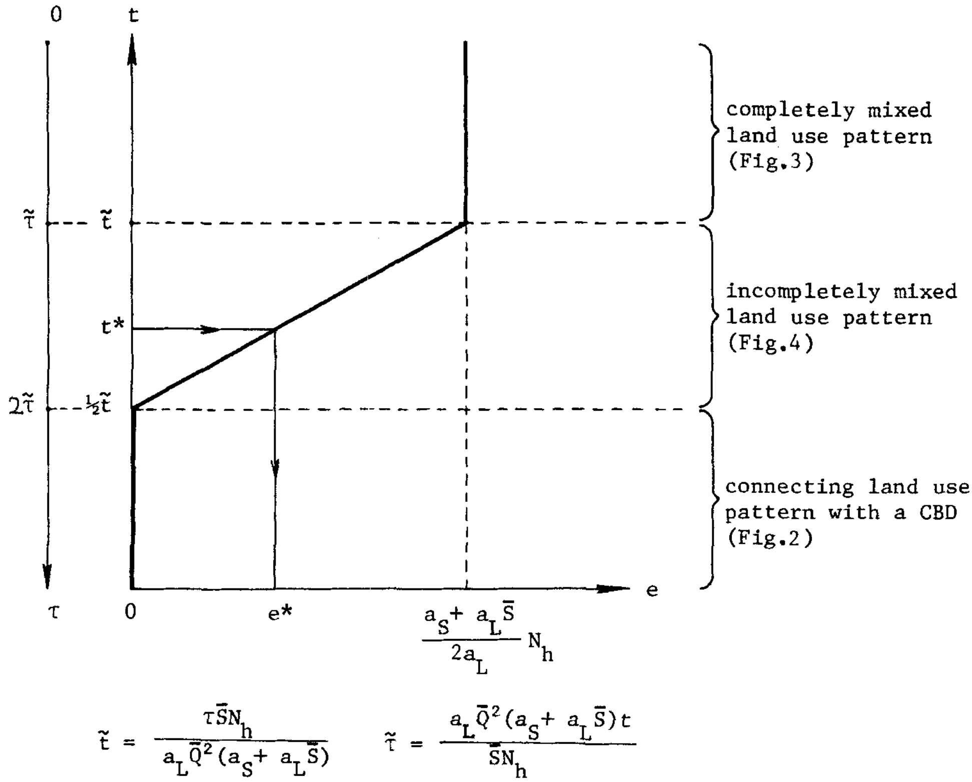

FIGURE 5: Equilibrium Land Use Patterns and Conditions on Parameters.

mixed land use pattern is observed at equilibrium and there is no dominant business district. The incompletely mixed land use pattern, then, can be considered as the intermediate equilibrium pattern between these two patterns. In this equilibrium pattern, the integrated district at the center is surrounded by business districts, which in turn are surrounded by residential areas. And if the value of t is specified at, say, t^*, then the size of integrated district, and therefore the whole structure of the urban configuration, can be determined.

5. CONCLUSIONS

In this article, we have proposed a model of nonmonocentric urban land use in which the location of neither employment nor residence is specified a priori. The model explicitly considers spatial interactions among activities; each firm's locational decision reflects its transactions with all other business firms, and each household's locational decision reflects its commuting trip to a business firm optimally chosen by the household. In this framework, we obtained three alternative equilibrium patterns of land use depending on the values of parameters in the model, especially on the commuting rate and the transaction rate.

From the findings of our analysis, it can be concluded that the monocentricity assumption is plausible only under special circumstances—that is, only when the commuting rate is relatively small and/or the transaction rate is considerably high. In all other cases, the city does not exhibit monocentricity. Moreover, in reality, the spatial forms of modern cities are too intricate to be analyzed in the framework based on the monocentricity assumption. Consequently, both from the viewpoints of theory and reality, the development of a nonmonocentric model like the one presented here seems to make a contribution to the theory of urban land use.

However, the model proposed in this article is quite simple, and a more satisfactory model is required. First, several strong assumptions should be relaxed; for example, the introduction of variable lot size for households and variable production level for business firms, and the replacement of input-output technology with a general production function. Second, the extension of our model into two-dimensional space is another important subject, since in the nonmonocentric framework the spatial structure of a city cannot be treated as if it were one-dimensional. Finally, questions may be addressed regarding the development of multicentric urban land use patterns and their effects on the spatial structure.

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