Economic Theory and the Cities

STUDIES IN URBAN ECONOMICS

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Economic Theory and the Cities

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Preface

In this book I present what I believe to be the most important theoretical topics in urban economics. Since urban economics is a rather diffuse field, any presentation is necessarily selective, reflecting personal tastes and opinions. Given that, I note on what basis I chose the material that is presented and developed.

First, the basic spatial model of a monocentric city is presented, since it lays the foundation for thinking about many of the topics in urban economics. The consideration of space and spatial proximity is one central feature of urban economics that distinguishes it from other branches of economics. The positive and negative externalities generated by activities locating in close spatial proximity are central to analysis of urban phenomena. However, in writing this book I have tried to maintain strong links between urban economics and recent developments in mainstream economic theory. This is reflected in the chapters that follow, which present models of aspects of the most important topics in urban economics—externalities, housing, transportation, local public finance, suburbanization, and community development. In these chapters, concepts from developments in economics over the last decade or so are woven into the traditional approaches to modeling these topics. Examples are the role of contracts in housing markets and community development; portfolio analysis in analyzing housing tenure choice and investment decisions; the time-inconsistency problem in formulating long-term economic relationships between communities, developers, and local governments; search in housing markets; and dynamic analysis in housing markets and traffic scheduling. The book ends with chapters on general equilibrium models of systems of cities, demonstrating how individual cities fit into an economy and interact with each other.

This book is written both as a reference book for people in the profession and for use as a graduate text. In this edition, a strong effort has been made to present the material at a level and in a style suitable for graduate students. The edition has greatly expanded the sections on housing and local public finance so these sections could be studied profitably by a broad range of graduate students. Recommended prerequisites are an undergraduate urban economics course and a year of graduate-level microeconomic theory. It is possible that the book can be used in very advanced undergraduate courses if the students are well versed in microeconomics and are quantitatively oriented.

Acknowledgments

I developed much of the material in this book in journal articles and for graduate urban economics courses at Queen's University, The University of Chicago, and Brown University. Most of the material in the first edition had been presented at some stage in its development in the Urban Economics Workshop at The University of Chicago. The comments of George Tolley as well as Charles Upton and the students in that workshop were instrumental in shaping my view of what is relevant and important in urban economics. I also benefited from many discussions with Peter Mieszkowski on a variety of issues in urban public economics.

The new material in this edition has been developed in my graduate courses at Brown University. The stimulating interaction with the students both in class and as they have worked on dissertations has helped refine much of the material. Parts of the material have also been presented in workshops at a variety of universities around the country and have benefited from those presentations.

I thank Marion Wathey for her skillful typing and my family for their support.

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Spatial Equilibrium and the Spatial Characteristics of a Simple City

In this chapter a simple model of a city is developed, with the following guidelines in mind. In specifying the model, we want to incorporate the basic features of the economic structure of cities. Thus the model should capture the essence of spatial interaction between producers and consumers in a city, and it should yield theoretical results that correspond to the basic empirical facts about cities. For example, the model should show that land rents, population density, and building heights decline with distance from the city center, as demonstrated in empirical work (e.g., Muth, 1969). It should be able to explain why higher-income people tend to live farther from city centers than lower-income people and why wages vary spatially within a city. When different cities with similar transportation technologies are compared, the level of rents, population density, and building heights should increase with city size. We also want a model of a city that can be adapted to enable us to analyze a system of cities and to describe equilibrium city sizes, factor movements, and trade patterns among the cities of an economy. Finally, the model should be consistent with the models and analyses of later chapters on housing, transportation, and public finance, which detail different aspects of urban living and are useful in analyzing specific urban problems.

In specifying the nature of cities in the model, the following assumptions are made. The economy consists of a flat featureless plain. Instead of the population spreading evenly over the plain, concentrations of population, or cities, form because it is assumed there are scale economies in production. Exploitation of these scale economies requires that there be concentrations of employment in production activities. These scale economies result from scale efficiencies in input markets, marketing, communications, transportation, and/or public service provision. Concentration of employment results in concentration of residences occupied by people who commute to the employment centers.

In cities most or all commerical activity occurs in the central business district (CBD), which is located in the central part of the city. This central location of all commerical activity results from businesses outbidding residents for this central land. The desire for businesses to be located together at the city center follows from several assumptions. First, if firms are located together, the advantages of scale economies may be more fully realized. Second, it is assumed that all goods produced in the city are shipped to a retailing and transport node at the very center of the city where they are sold to city residents and exported to other cities. This node could be a railway station, trucking terminal, or harbor (in a semicircular city). Firms minimize the costs of shipping goods to the node by locating around the node. Finally, we note that, because the business district is at the center of the city, the total costs of commuting to work for all residents are minimized relative to the business district's being at a noncentral location.[1]

Surrounding the CBD is the residential sector where all city residents live. From their home sites residents commute to the city center and then disperse to their work sites. It is this feature of most or all residents commuting to work in the CBD that distinguishes a simple city from more complicated cities, where only part of the city's labor force commutes to the CBD. At various points in the book the impact of non-CBD employment on the basic results of the model will be considered.

Finally, in a stable equilibrium solution, both the CBD and the total city will be symmetric circles. As shown later, this result follows when there is only one business district because the plain on which cities are located is featureless.

In the first section of this chapter the residential sector of the city is examined. Equilibrium of a household in space is analyzed. Then, building

[1] Being at the center of a circle minimizes the distance involved in traveling to all points in the circle. Since our city will be a circle, the central location of the CBD minimizes total commuting costs, given there can be only one business district. With two or more business districts, this proposition is no longer correct.

upon the properties of a household's equilibrium, we study long-run equilibrium in the housing and land markets. Finally, aggregate demand and supply relationships in the residential sector are derived. Throughout, the general concepts developed are illustrated with a simple example using specific functional forms. In the second section of the chapter, the commercial sector of the city is examined. Building upon the individual producer's profit maximization problem, aggregate relationships describing the commercial sector's use of labor, capital, and land are developed. These aggregate relationships will be used to determine equilibrium levels of employment and prices in Chapter 2.

1. THE RESIDENTIAL SECTOR

1.1 Consumer Residential Choice and Equilibrium in the Residential Sector

Residents in the city maximize utility defined over market goods and amenities subject to a budget constraint and the amenity choices facing them. Market goods are the city's own traded good x produced in the CBD, the city's import good z, and housing services h, which are rented from housing producers. The prices of the traded goods, p_x and p_z, do not vary within the city since these goods are all purchased from the same market at the center of the city. The rental price of housing may vary spatially; and, in fact, housing and housing prices are distinct items in the model.

Housing represents both a consumer good and a particular spatial location in the city. Associated with each spatial location in the residential sector is a level of amenities consumed by residents. The only amenity I consider in this chapter is leisure consumption, which is directly related, through commuting times, to access to the CBD. The rental on housing implicitly prices both housing services and access, or leisure consumption; and thus the unit price of housing $p(u)$ will vary spatially as leisure varies. This amenity formulation is perfectly general and can be expanded to include a vector of goods such as park and recreational services and clean air (see Chapter 4).[2]

With respect to leisure, we assume that residents work a fixed number of hours. Leisure is the fixed number of nonworking hours T less time spent commuting. We assume that the time it takes to commute a *unit* distance (there and back) to work is t; and t is the same everywhere in the city. (This

[2] I first came across the general formulation in Hartwick (1971). Diamond (1976) also used a similar formulation.

assumption implies there is no congestion; or as the number of commuters accumulates as we approach the CBD, travel speeds are unchanged.) Therefore, a consumer at distance u from the city center has leisure consumption $e(u)$ equal to

$$e(u) = T - tu.$$

Note that time costs are the only form of commuting costs in this chapter.[3] It is a straightforward exercise to add out-of-pocket commuting costs (e.g., automobile operating costs) to the model through the budget constraint. Situations where several amenities or costs vary with distance from the CBD are analyzed in Chapter 3.

Given these assumptions, I can now formally state the consumer optimization problem. Where $V(u)$ is utility at location u and y is income, the consumer

$$\max_{\text{w.r.t. } x,z,h,e,u} \quad V(u) = V'(x(u), z(u), h(u), e(u)) \tag{1.1}$$

subject to

$$y - p_x x(u) - p_z z(u) - p(u)h(u) = 0,$$
$$T - e(u) - tu = 0.$$

For consumers this is essentially a simultaneous two-stage maximization problem. They must pick an optimal location in space, given the spatial set of amenities and housing prices; and at the optimal location, they must choose an optimal consumption bundle. For consumers at location u, their optimal consumption bundle is chosen according to the budget constraint and the usual first-order conditions equating price ratios with marginal rates of substitution in consumption. Given these conditions and assuming that V' is a regular utility function,[4] we can then specify individual consumer demand equations for all market goods as a function of income, all output prices, and leisure. For example, for housing

$$h(u) = h(y, p(u), p_x, p_z, e(u)), \tag{1.2}$$

where h is increasing in y and decreasing in $p(u)$.

The question we are primarily concerned with is how consumers come to choose a particular u or distance from the city center. Maximizing Equation (1.1) with respect to $e(u)$ and u yields the first-order conditions that

[3] I first came across this formulation in Beckmann (1974).

[4] The utility function V' should be a continuous, nondecreasing, and strictly quasi-concave function.

$\partial V'/\partial e(u) - \gamma = 0$ and $-\lambda h(u)[\partial p(u)/\partial u] - \gamma t = 0$ where γ and λ are Lagrange multipliers and are, respectively, the marginal utility of leisure and that of income. Combining to solve out γ yields the condition that holds when consumers are at their optimal locations

$$h(u)\frac{\partial p(u)}{\partial u} = -\frac{\partial V'/\partial e(u)}{\lambda} t \equiv -p_e(u)t. \tag{1.3}$$

The term $p_e(u)$ is the monetized value of the marginal utility of leisure, where we have defined

$$p_e(u) = [\partial V'/\partial e(u)]/\lambda.$$

This term measures the marginal evaluation of leisure, which is the opportunity cost of travel time.

At their optimal locations, if consumers move an infinitesimal distance farther from the city center, they experience a loss in leisure. The value of this lost leisure is the marginal evaluation of leisure $p_e(u)$ multiplied by the reduction in leisure $-t$. Equation (1.3) states that they are exactly compensated for this lost leisure by reduced housing costs $h(u)[\partial p(u)/\partial u]$, such that utility is unchanged.[5] That is, at an optimal location they cannot improve their welfare by moving. This implies that $\partial p(u)/\partial u < 0$, where this decline in housing rents is necessary to compensate consumers for lost leisure time as they move farther from the city center. Otherwise, consumers could not be induced to live farther from the center and we could not have an equilibrium set of locations. If housing rents rose or stayed constant as consumers moved away from the city center, a consumer would always be better off moving inward since leisure would be increased with unchanged or lower housing costs.

Equation (1.3) describes a relationship between equilibrium housing rents and distance that must hold for an individual household to be in equilibrium. The next step is to derive the properties of the set of equilibrium housing prices that occurs along a ray from the city center. This set of prices is called the rent gradient. The rent gradient is defined by its height and slope, or by the level of prices at each distance from the CBD and the change in these prices as distance changes. In the next section are derived these properties and in the following section I show how the rent gradient must be consistent with equilibrium in the residential housing and land markets.

[5] That is, at the optimal location we are at a stationary point where infinitesimal changes in location bring no utility changes, or $dV du = 0$, given the constraints of the problem. This formulation involves certain continuity and smoothness assumptions about how utility varies over space, as will become clearer later.

The Rent Gradient with Identical Consumers

If all residents in a city have identical incomes and tastes, then deriving the properties of the residential rent gradient is straightforward. If consumers are identical, in a stable spatial equilibrium all residents must have the same utility level at their different locations. Otherwise residents in locations with lower utility levels will bid for locations with higher utility levels, driving up prices at those locations and/or causing spatial movements. The situation is only stable when all identical residents are equally well off.

To derive the properties of the rent gradient, we introduce the concept of an indirect utility function. As indicated earlier, for a consumer at location u, there exists a set of demand equations for market goods where demand is a function of income, prices, and leisure. When these demand equations for market goods are substituted into the direct utility function, utility indirectly becomes a function of income, prices, and leisure. We may then define the indirect utility function, or

$$V = V(y, p(u), p_x, p_z, e(u)), \tag{1.4}$$

where V is increasing in y and e, decreasing in prices, and homogeneous of degree zero in income and prices. One interesting property of the indirect utility function utilized at various points in the book is that the demand for housing (and similarly for other goods) may be represented as[6]

$$h(u) = -\frac{\partial V/\partial p(u)}{\partial V/\partial y}. \tag{1.5}$$

Equation (1.5) follows from Roy's identity.

With identical residents the rent gradient must be such that utility in Equation (1.4) is the same everywhere in the city. Therefore, housing prices must vary such that $dV/du = 0$. Accordingly I could differentiate Equation (1.4) and do appropriate substitutions to find the slope of the rent gradient.[7] Alternatively note that since Equation (1.3) specifies a relationship between actual housing prices and distance that must hold for individuals to be in

[6] An intuitive explanation why Equation (1.5) holds is simple. The term $-\partial V/\partial p(u)$ is the marginal utility obtained from a dollar decline in housing prices. This equals the marginal utility of a dollar ($\partial V/\partial y$) multiplied by the change in dollars available to the consumer, which equals the number of housing units $h(u)$ multiplied by the dollar change in price (or 1). Rearranging terms yields (1.5).

[7] We differentiate Equation (1.4), set $dV = 0$, divide by $\partial V/\partial y$, and substitute in Equation (1.5) and the expression for $p_e(u)$. Rearranging terms yields Equation (1.6). Equation (1.4) can also be used to derive the consumer's spatial equilibrium condition where, at a utility-maximizing location, $\partial V/\partial u = 0$ and $\partial^2 V/\partial u^2 \leq 0$.

THIS IS REVERSE ORDER OF SOLUTION FROM P 4

equilibrium, it must also define the slope of the rent gradient that holds in a stable-market equilibrium. Rearranging Equation (1.3) yields the slope of the rent gradient

$$\partial p(u)/\partial u = -h(u)^{-1} p_e(u) t < 0. \tag{1.6}$$

We can solve for the height of the rent gradient at any point using the indirect utility function. To demonstrate this examine Figure 1.1a, in which a residential rent gradient between u_0 and u_1 is illustrated. The CBD and city radii are represented by u_0 and u_1, respectively; thus the residential area lies between u_0 and u_1. The basic reference point on the gradient is at the city edge u_1. Consumers at the city edge have utility $V(u_1)$ defined in Equation (1.4) by leisure $e(u_1)$ given commuting time tu_1, by income and traded good prices, and by the known price of housing at u_1. The price of housing at u_1, or $p(u_1)$, equals the known price received from producing housing on land at u_1, which borders on agricultural land. [As we shall see later, $p(u_1)$ is determined by known agricultural rents and the price of capital.] Given that all consumers have identical tastes, $V(u_1)$ defines utility throughout the city. Then from Equation (1.4), for any u_i, given the known values of V and $e(u_i)$, we should be able to solve for $p(u_i)$, the height of the rent gradient at that point.

To do so we invert Equation (1.4) to get

$$p(u) = p(y, p_x, p_z, e(u), V). \tag{1.7}$$

Substituting in for $V = V(u_1) = V(y, p_x, p_z, e(u_1), p(u_1))$ and rearranging, we get

$$p(u) = \tilde{p}(y, p_x, p_z, p(u_1), u_1, u; t, T). \tag{1.7a}$$

While this discussion demonstrates a method for deriving the equilibrium rent gradient, we still need to know how u_1 and hence $V(u_1)$ are determined. I turn

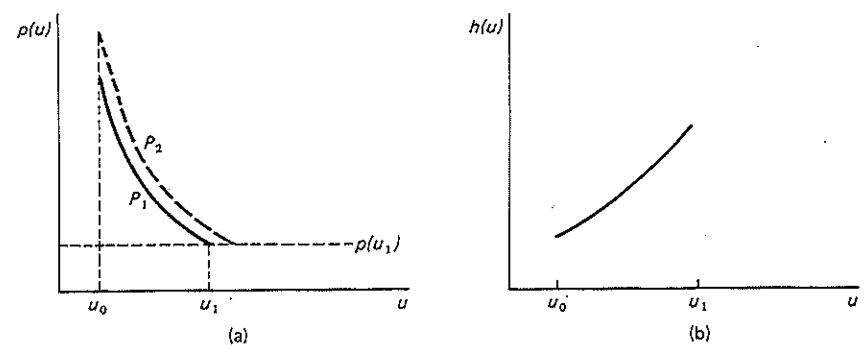

FIGURE 1.1 Rent gradients: identical consumers.

to that topic in the next section where I examine equilibrium in the housing and land markets, but first I comment on several other properties associated with the equilibrium rent gradient.

So far I have described the equilibrium change in rents for consumers moving along a ray from the city center. To determine the equilibrium pattern of rents throughout the city, circumferential movements by consumers must also be considered. Equilibrium with respect to circumferential movements occurs when along all rays from the CBD the rent gradient is the same as P_1 in Figure 1.1a, so that at a given radius from the city center all rents on that circumference are equalized. Since commuting costs to the CBD are the same from any point on that circumference, consumers will then have no incentive to switch locations circumferentially. For this reason cities on a flat featureless plain must be circular or else people will bid to fill out a circle.

Finally we note that, because $p(u)$ declines, for discrete spatial moves the consumer's housing consumption will change in response to the changing price. Since, with identical consumers, utility remains constant as locations and prices change, the housing demand response is described by the Hicks pure substitution effect where $\partial h(u)/\partial p(u) < 0$. Therefore housing consumption increases with distance as price declines. This relation is illustrated in Figure 1.1b.

This increase in housing consumption is also the reason the rent gradient is pictured as convex.[8] Convexity implies that $\partial p(u)/\partial u$ in Equation (1.6) declines with distance, or $\partial^2 p(u)/\partial u^2 > 0$. This decline in $\partial p(u)/\partial u$ in Equation (1.6) occurs because h is increasing as u increases, which causes the right-hand side of Equation (1.6) to decline, providing $p_e(u)$ does not vary much with discrete spatial moves. Alternatively stated, with convexity, as we move out the rent gradient and leisure declines, the approximately equal compensating decline in housing expenditures at each point, which is $h(u)$ $\partial p(u)/\partial u$, is achieved with smaller and smaller changes in unit prices, $\partial p(u)/\partial u$, given that housing consumption $h(u)$ is continuously increasing.

The Rent Gradient and Housing Market Equilibrium

The height and length of an equilibrium rent gradient like that pictured in Figure 1.1a must be consistent with conditions defining housing market equilibrium in the residential sector of the city. This equilibrium is determined by demand and supply conditions in the housing market and in factor markets underlying the housing market. Although most of the properties that define a stable equilibrium in the housing market are quite obvious, I state them here

[8] This convexity is not necessary. It implies that $\partial^2 p(u)/\partial u^2 = (-\partial h/\partial p)(\partial p/\partial u)^2 h^{-1} + (\partial p_e(u)/\partial e)t^2 h^{-1} > 0$. If $\partial p_e(u)/\partial e$ is small (or positive), this condition is met.

for emphasis because it is important to have them firmly in mind for the discussion that follows and when we analyze more sophisticated situations in the housing and land markets in Chapter 4.[9] Since the housing market equilibrium cannot be entirely isolated from conditions in factor markets, I also briefly describe the supply of housing here. It is analyzed in Section 1.2.

Consumers rent housing from housing producers.[10] Housing producers produce housing according to the usual profit-maximization conditions with rented capital and land. Capital is perfectly malleable and mobile and is rented by housing producers at a fixed price in local or national capital markets. Land is owned either by a class of people called rentiers or collectively by city residents through the operation and management of the city government (see Chapter 2). I have distinguished four groups of people—consumers, producers, capital owners, and rentiers. However, the specification is general and we can collapse these people into three, two, or even one group. Consumers could produce their own housing by renting inputs; rentiers or capital owners could produce the housing; consumers could be the rentiers and own their own land; and so on.

Given this situation, four conditions define a stable equilibrium in the housing market. Although the conditions are specific to the housing market, with the land market being discussed separately in Section 1.2, since the two markets are not independent, in some of the conditions the land market is referred to.

1. In equilibrium, on the supply side, suppliers of housing rent to the consumer who is willing to pay the most for that housing; and thus housing producers have no incentive to switch customers or tenants. In Figure 1.1a, since all consumers have equal utility along the rent gradient P_1, at any location no other consumer would be willing to pay more for that housing than the current resident.
2. On the demand side, each consumer rents the housing and location that maximizes utility, given the equilibrium set of prices; and thus no renter has an incentive to move. In Figure 1.1a, since all residents have equal utility, given P_1, consumers cannot improve their welfare by bidding away, and thus raising, the price of the housing of another resident.
3. The boundaries of the residential area are the CBD radius u_0 and the city radius u_1. At these boundaries the price of residential housing equals its

[9] For an analysis of the attainment of residential spatial equilibrium, the reader should consult Alonso (1964), Chapters 4 and 5 and Appendix A).

[10] We assume consumers rent rather than purchase since that fits in with the structure of a single-period model and comparative statics. If consumers own housing, we would have to employ a multiperiod model, use a wealth constraint, and consider capital gains and losses.

opportunity cost, or the cost of producing housing on land in the alternative competing use at the boundary. If the residential price exceeds [is less than] that cost, housing producers and land owners have an incentive to increase [decrease] the residential area. Since the price of capital is everywhere the same, we will show that higher [lower] housing prices directly reflect higher [lower] land prices. Then, for example, if the cost of producing housing on land in the alternative use at u_1 is lower than the current price of housing there, this means the price of land in the alternative use is lower than the residential price. Landowners will individually profit by increasing the allocation of land to urban use, until land prices and hence housing costs and prices are equalized at the border of competing uses. Then the city's spatial area will be stable.

4. All housing and locations supplied are rented; and given u_0, u_1, and the rent gradient, all residents consume their desired level of housing. That is, demand equals supply and there are no holes in the city or misplaced residents. If the city population were to increase, in aggregate more housing and hence more land would be demanded. The outer bound of the city u_1 would increase, and to accommodate the new population for the same $p(u_1)$, the rent gradient would shift up to, say, P_2 in Figure 1.1a for the same income and prices. The fact that the rent gradient shifts up is proved formally in Section 1.3. The degree to which P_2 shifts up depends on both the increase in population and the partially offsetting decline in per person housing and derived land demands at each point as housing prices rise. Note that with the increase in population and rise in P_2, consumers will all be worse off if their incomes are unchanged. For example, in the indirect utility function, for the person on the new city edge (which is our reference point for defining utility levels in the city), leisure, or $e(u_1)$, declines while all other variables are unchanged.

The Rent Gradient and Nonidentical Consumers

To derive the equilibrium rent gradient when consumers are not identical is more complicated. We assume that consumers differ only by income. To solve for the gradient, the concept of a bid rent function is utilized. A bid rent function describes what unit rents a particular consumer would be willing to pay for housing services in different locations, such that he is indifferent among these locations.

To define a bid rent function we use Equation (1.7), where $p(u)$ is replaced by $p^0(u)$ so

$$p^0(u) = p(y, p_x, p_z, e(u), V). \tag{1.7b}$$

Equations (1.7) and (1.7b) are both bid rent functions. However, $p^0(u)$ is a hypothetical bid price in a Walrasian auctioneering process, while $p(u)$ is the equilibrium price bid at which transactions occur. Bids vary with u and V in Equation (1.7b).

To determine the properties of Equation (1.7b), we differentiate the indirect utility function, holding income and other prices fixed. If we set $dV = 0$, divide by the marginal utility of income, and substitute in Equation (1.5) and the expression for $p_e(u)$, the result is

$$\partial p^0(u)/\partial u = -h(u)^{-1} p_e(u)t. \tag{1.8}$$

Equation (1.8) defines the slope of a bid rent curve, and it indicates how a person's bid rent will vary along a ray from the city center. A set of bid rent curves is pictured in Figure 1.2a. The height of the bid rent curves is defined by the utility level in (1.7b), where along any curve utility is fixed and the curves shift up [down] as utility falls [rises]. This property is expressed by the negative relationship in the indirect utility function between V and $p^0(u)$ for $e(u)$ fixed. Bid rent curves for the same individual do not cross, just as indifference curves do not cross.

In general, bid rent curves and in particular the slope of bid rent curves vary as income varies among individuals. As income changes, $h(u)$ and $p_e(u)$ will vary; hence, at each location u, from Equation (1.8), the slope of the bid curves should either increase or decrease. I choose to illustrate an equilibrium where the slope decreases as income increases. In Equation (1.8) this means that the quantity of housing consumed $h(u)$ increases *relative* to the marginal evaluation of leisure $p_e(u)$ as income increases. This assumption is discussed in detail after I illustrate the nature of an equilibrium rent gradient.

For the illustration, I first rank *groups* of equal-income consumers by the steepness of their bid rent curves. The steepest is placed near the CBD and the least steep near the city edge. Figure 1.2b shows four bid rent curves labeled a, b, c, and d, one for each of the four different groups of consumers. As just

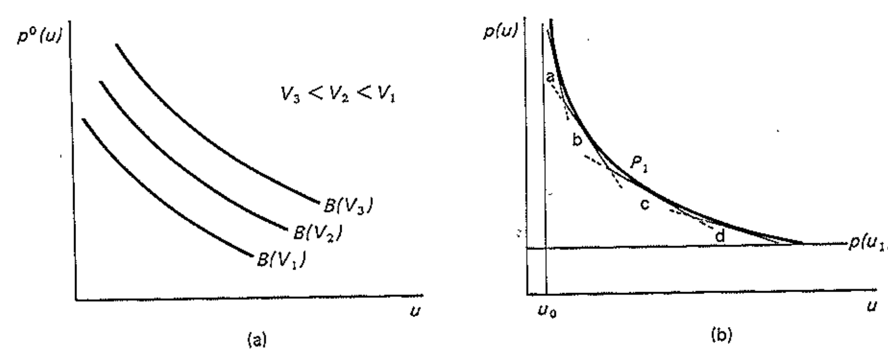

FIGURE 1.2 Bid rents and gradients for nonidentical consumers.

assumed, the steepest bid rent curve belongs to the lowest-income consumers. The equilibrium rent gradient P_1 is an *envelope* of bid rent curves and is composed of a segment for each income group where within each segment the bid rent curve and rent gradient are coincident. Each point on the gradient represents how much people have to pay to bid their house and location away from other users in their or other income groups. At the points of junction of the rent gradient segments, the dashed extensions of the bid rent curves that do not lie on the rent gradient represent what people in a particular income group would be willing to pay to live in another part of the city such that their utility is unchanged. The properties of the equilibrium rent gradient are as follows.

From Equation (1.8), the slope of a bid rent function, which is also the slope of the rent gradient, specifies the same price–distance relationship as Equation (1.3). This implies that along the equilibrium rent gradient consumers are at their utility-maximizing location.

The ordering of people by the steepness of their bid rent curves is necessary for stability and satisfies the two market equilibrium conditions. (1) Producers are renting to the highest bidders. For example, we can see by comparing the rent gradient and the dashed extensions of the bid rent curves that type b consumers would not outbid type a consumers interior to the point of junction of their equilibrium bid rent curves and rent gradient segments. (2) Consumers are at their utility-maximizing location. For example, type b consumers would be worse off, or on a higher bid rent curve, if they paid the prices type a people paid. Any other spatial configuration, such as b people living interior to type a people, would be unstable, since then type a people would be willing to pay higher prices than b people to live in b's segment of the city. (We can see this by redrawing Figure 1.2b with less steep bid rent curves nearest the city center.)

In Figure 1.2b the height of the rent gradient, the size of the residential area, and the size of the segments within which each income group lives are such that all people are housed, all land and housing is rented in the city, and all people consume their desired level of housing given their location and the prices on the rent gradient. As with a city of identical consumers, an increase in population will shift up the rent gradient and extend u_1.

If the underlying reasons for how the slopes of bid rent curves vary are examined, the ordering of consumers by the steepness of their bid curves turns out to be intuitively appealing. If slopes decrease at each point as income increases, this means, from Equation (1.8) where $\partial p^0(u)/\partial u < 0$, that $d(\partial p^0(u)/\partial u)/dy > 0$. Differentiating Equation (1.8) with respect to y for u fixed, we know

$$d(\partial p^0(u)/\partial u)/dy = h(u)^{-1} y^{-1} p_e(u) t[\eta_{h,y} - \eta_{p_e,y}] \gtreqless 0, \qquad (1.9)$$

where $\eta_{p_e,y}$ is the income elasticity of the marginal evaluation of leisure and

$\eta_{h,y}$ is the income elasticity of demand for housing. Whether Equation (1.9) is positive or negative depends on whether the expression in brackets is positive or negative, and hence it depends on whether $\eta_{h,y} \gtreqless \eta_{p_e,y}$. For the assumption made earlier that the slopes of rent curves decrease as income increases, or $d(\partial p^0(u)/\partial u)/dy > 0$, $\eta_{h,y} > \eta_{p_e,y}$. From the definition of elasticities, this means that for a 1% increase in income, the percentage increase in housing is greater than the percentage increase (if any) in the marginal evaluation of leisure. In these circumstances lower-income people live closest to the CBD and have the most leisure.

This makes sense since lower-income people who end up nearest the CBD are those people who value leisure the most *relative* to housing. Thus they are the people who are willing to pay the highest price per unit of housing near the CBD to housing producers. A simple numerical example can be used to illustrate this point. Suppose higher- and lower-income people would consume 10 and 5 units, respectively, of housing given current prices at a location u. At that location lower-income people would be willing to pay \$5 per week more (indirectly through higher housing prices and hence payments) to move slightly closer to the CBD and have a unit increase in leisure. Suppose higher-income people are only willing to pay \$9 a week more (through larger housing payments) for a unit increase in leisure. Then the percentage by which higher-income people's housing consumption is larger than lower-income people's is greater than the percentage by which higher-income people's marginal evaluation of leisure is larger (i.e., 100% > 80%). This means that the increase in price bid per unit of housing to move closer to the CBD for higher-income people is less than for lower-income people (i.e., \$9/10 < \$5/5) and housing producers will accept the higher bids of lower-income people for high-access land. On the other hand, if higher-income people are willing to pay \$11 for a unit increase in leisure, they will outbid lower-income people for high-access land (i.e., \$11/10 > \$5/5). The problem for higher-income people trying to live next to the CBD is that, even if they are willing to pay more than lower-income people in absolute terms for increased leisure, their percentage difference in housing consumption relative to lower-income people cannot be greater than their percentage difference in leisure evaluation or the effect of their greater leisure evaluation on unit housing prices is dissipated through their higher housing consumption. This type of argument can also be used to explain why high- or low-income people tend to live in more polluted or higher-crime areas.

The assumption that $d(\partial p^0(u)/\partial u)\,\partial y > 0$ is a rather arbitrary theoretical assumption. This assumption was made in the foregoing discussion and is made in the literature since it yields results consistent with the empirical observation that higher-income people tend to live farther from the city center in the United States. However, that empirical phenomenon could be explained on other grounds and in more sophisticated models not considered until later in the book. For example, in the housing filtering-down models in Chapter 6, higher-income people tend to live in the newest housing, which, given the age and development of American cities, generally is built on the outskirts of cities. In Chapter 10, we note that higher-income people have a fiscal incentive to suburbanize and hence to move farther away from the city center than lower-income people.

Non-CBD Local Employment In deriving these rent gradients it has been assumed that all people commute to the CBD. This need not be so. Suppose in Figure 1.2b that on the segment of the P_1 rent gradient where type a people live, some of the type a people work in a grocery store in that area. Providing that most people still work in the CBD, the shape of the rent gradient is still determined by the same equilibrium conditions for CBD commuters and is unchanged. That is, their bidding determines the competitive price of land. Identical type a people who work locally pay the same market rents as commuters but have more leisure. To maintain equilibrium in labor markets and choice of occupation, the local wages of noncommuters will be lower than those of commuters by the value of their increased leisure. If, however, the ratio of local to CBD workers becomes too large, then the gradient will change. For example, if beyond a certain point no one commutes to the CBD, the rent gradient would be radically different.

An Illustration of Housing Rent Gradients in a Simple City

It is useful to illustrate the foregoing discussion with a simple example using specific functional forms. This example and the specific functional forms will also be used later to derive explicit aggregate relationships for a city. Consumers maximize a logarithmic linear utility function subject to a budget constraint and a leisure constraint.

Therefore, the consumer maximization problem is to

$$\max_{\text{w.r.t.}\,x,z,h,e,u} \quad V = A'x(u)^a z(u)^b h(u)^c e(u)^d, \qquad a+b+c=f \tag{1.1a}$$

subject to

$$y - p_x x(u) - p_z z(u) - p(u)h(u) = 0,$$
$$T - e(u) - tu = 0.$$

The first-order conditions are $aV/x(u) - \lambda p_x = 0$, $bV/z(u) - \lambda p_z = 0$, $cV/h(u) - \lambda p(u) = 0$, $dV/e - \gamma = 0$, and $-\lambda h\,(\partial p(u)/\partial u) - \gamma t = 0$ where λ is the marginal utility of income and γ is the marginal utility of leisure time.

Substituting the first-order conditions with respect to the consumption of market goods into the budget constraint, we can get demand equations. For example, from the first-order conditions we know that $p_x x(u) = (a/c)p(u)h(u)$ and $p_z z(u) = (b/c)p(u)h(u)$. Substituting these in the budget constraint and solving, we find

$$h(u) = (c/f)yp(u)^{-1}. \tag{1.2a}$$

The other demand equations are $x(u) = (a/f)yp_x^{-1}$ and $z(u) = (b/f)yp_z^{-1}$.

By combining and arranging the first-order conditions for e and u we find

$$h(u)\frac{\partial p(u)}{\partial u} = -t\frac{Vd/e(u)}{\lambda} = -tp_e(u). \tag{1.3a}$$

As a consumer moves farther away from the city center, the value of lost leisure is compensated by reduced housing costs. By substituting $e(u) = T - tu$ and $\lambda = cV/[p(u)h(u)]$ from the first-order conditions into Equation (1.3a), we can rewrite this equation to get the slope of bid rent curves and rent gradients.

$$\partial p(u)/\partial u = -(td/c)p(u)(T - tu)^{-1}. \tag{1.6a}$$

The alternative way to derive Equation (1.6a) is to use the indirect utility function. To derive the indirect utility function we substitute the consumer demand equations and $e = T - tu$ into the direct utility function to get

$$V = Ay^f p_x^{-a} p_z^{-b} p(u)^{-c}(T - tu)^d, \tag{1.4a}$$

where $A = A'(a/f)^a(b/f)^b(c/f)^c$. Maximizing V in (1.5a) with respect to u yields Equation (1.6a).

In this illustrative example, the slope of the bid rent curve and rent gradient in Equation (1.6a) is independent of income and hence holds for all income levels. Therefore, Equation (1.6a) also describes the slope of the rent gradient in a city where people have either equal incomes or differing incomes. To find the height of the rent gradient we write Equation (1.6a) in logarithmic form, integrate, and then take antilogarithms to get

$$p(u) = C_0(T - tu)^{d/c},$$

where C_0 is the constant of integration.

The most general way to evaluate C_0 for a particular city is the following. We know the opportunity cost of land in agriculture (which can be zero) that the city must pay to get land at the border of the city. Therefore, we can determine urban housing prices at the city edge (see Section 1.2), or $p(u_1)$ in Figure 1.1b. Evaluating at the city edge, we have $p(u_1) = C_0(T - tu_1)^{d/c}$.

Solving for C_0 and substituting into the rent gradient expression, we get

$$p(u) = p(u_1)(T - tu_1)^{-d/c}(T - tu)^{d/c}. \quad (1.10)$$

This is the residential rent gradient for a city of either identical or multi-income people. The height of this gradient at any location is determined by the housing rent at the city edge, the endogenous spatial size of the city as measured by u_1 (which is solved for later in the chapter), and the parameters of the model. A rent gradient is illustrated in Figure 1.1b or 1.2b by P_1.

Another common way to evaluate C_0 is to assume that (1) all people have identical tastes and income and hence in equilibrium have identical utility levels, and (2) the utility level in the city is fixed at a level $\dot{V}$, which is given by an infinitely elastic supply curve of labor to the city at utility level $\dot{V}$ (see Chapter 2). If we know V, then by rearranging Equation (1.4a) we know

$$p(u) = (\dot{V})^{-1/c} A^{1/c} y^{f/c} p_x^{-a/c} p_z^{-b/c} (T - tu)^{d/c}$$

or

$$C_0 = (\dot{V})^{-1/c} A^{1/c} y^{f/c} p_x^{-a/c} p_z^{-b/c}.$$

1.2 Production of Housing

So far, we have investigated and illustrated consumer spatial equilibrium, housing rent gradients, and equilibrium in the housing market. We still have to investigate fully the supply side of housing, land rent gradients, and equilibrium in the land and capital markets.

Housing is produced under constant returns to scale with land l and capital k where

$$h(u) = h(k(u), l(u)). \quad (1.11)$$

Producers seek to maximize profits $\pi(u) = p(u)h(u) - p_k k(u) - p_l(u)l(u)$ where p_k is the spatially invariant price of captial and $p_l(u)$ is the price of land at location u. At a given location, inputs are employed according to the usual first-order conditions describing marginal productivity conditions, or $p_l = p(u)(\partial h/\partial l)$ and $p_k = p(u)(\partial h/\partial k)$. It often is convenient to alternatively describe production technology by the unit cost function where the unit cost of production $c = p(p_k, p_l)$. The function p is linear homogeneous, increasing in input prices and subsumes efficient factor usage by the firm.[11] If there is perfect

[11] In general, for the existence of unit cost functions with or without scale as an argument, h should be nondecreasing in its arguments, a right-continuous function, and quasi-concave.

competition, so that retail price equals unit production costs, the unit cost relationship can be expressed as

$$p(u) = p(p_k, p_l). \quad (1.12)$$

In the previous section we saw that housing prices must vary spatially to maintain consumer equilibrium. This means the gross revenue from producing a unit of housing will vary spatially. Hence, in addition to choosing optimal input combinations at any location, producers are concerned with choosing a profit-maximizing location. A producer's profit-maximizing location is one where $\partial\pi/\partial u = 0$ or

$$h(u)[\partial p(u)/\partial u] = l(u)[\partial p_l(u)/\partial u]. \quad (1.13)$$

Equation (1.13) states that when producers move an infinitesimal distance from their optimal location, their change in land costs exactly equal their change in housing revenue. That is, given that their original location is optimal, they cannot be made better off by moving.

The next step in the analysis is to derive the characteristics of the set of equilibrium land prices, which is the land rent gradient. First, note that, if producers are identical in terms of their technology and entrepreneurial ability, profits from building housing must be everywhere equal. If housing is a competitive industry, profits are zero. Zero profits are realized by housing producers bidding up [down] the rent paid on land in locations where nonequilibrium profits are positive [negative] until profits are zero. Zero profits also imply that unit costs in Equation (1.12) must always vary through land costs to equal output prices. Given these assumptions, the slope of the land rent gradient may be found in several ways. We can differentiate the profit function and rearrange terms, given $d\pi/du = 0$; or we can differentiate the unit cost function and, after appropriate substitutions, rearrange terms to get the slope.[12] Alternatively, since Equation (1.13) specifies a land rent–distance relationship that must hold for individual producers to be in equilibrium, it must also indicate the slope of the land rent gradient that must hold in a stable-market equilibrium. Rearranging Equation (1.13) yields the slope

$$\partial p_l(u)/\partial u = h(u)l(u)^{-1}[\partial p(u)/\partial u]. \quad (1.14)$$

[12] Differentiating the unit cost function yields $\partial p(u)/\partial u = \partial p/\partial p_l(u)\, \partial p_l(u)/\partial u$. To interpret this condition we note a useful property of unit cost functions. From Shephard's lemma (Diewert, 1974) $k(u)/h(u) = \partial p/\partial p_k$ and $l(u)/h(u) = \partial p/\partial p_l(u)$; or the derivative of the unit cost function equals the per unit demand for the respective factor. This may be explained intuitively as follows. The term $\partial p/\partial p_k$ is the increase in unit costs if the price of capital rises by \$1. This increase in unit costs equals the number of units of capital employed multiplied by \$1 divided by the number of housing units, or it equals $k(u)/h(u)$. Substituting the equation for $\partial p/\partial p_l(u)$ into the spatial equilibrium condition on unit costs yields Equation (1.14).

The unit cost function can be used to solve for the height of the rent gradient at each point, because given $p(u_i)$ and p_k, Equation (1.12) can be solved for $p_l(u_i)$. Inverting Equation (1.12), substituting in Equation (1.7a), and rearranging we get

$$p_l(u) = p_l(y, p_x, p_z, p(u_1), p_k, u_1, u; t, T). \quad (1.15)$$

A land rent gradient is pictured in Figure 1.3 and is consistent with equilibrium in land markets. Equilibrium in land markets satisfies the four types of conditions listed for housing. All land supplied will be rented at a nonnegative price, or there are no holes or vacant areas in the city; and all housing producers will rent their desired quantity of land given prices (condition 4). Land rents at the borders of competing uses, such as agricultural or commercial uses, will be equalized (condition 3). Landowners in equilibrium receive the maximum rent anyone is willing to pay for their land (condition 1). Demanders of land, such as housing producers, receive their maximum possible profits (zero) at their equilibrium land site, relative to other sites that they could rent and build housing on (condition 2).

Spatial Characteristics Implied by the Equilibrium Rent Gradient

Several important results follow directly from Equation (1.14). In Equation (1.3) I showed that changes in housing costs, $[\partial p(u)/\partial u]h(u)$, due to infinitesimal spatial moves exactly equal the change in leisure multiplied by the marginal evaluation of leisure, $-tp_e(u)$. From Equation (1.14) we can see that changes in land rents paid by housing producers exactly equal changes in housing costs, and hence they also equal the value of marginal leisure losses.

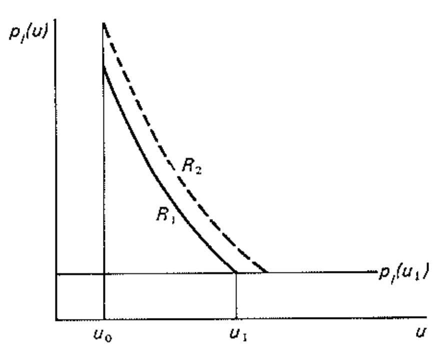

FIGURE 1.3 Land rent gradients.

That is,[13]

$$l(u)\frac{\partial p_l(u)}{\partial u} = h(u)\frac{\partial p(u)}{\partial u} = -p_e(u)t.$$

Second, Equation (1.14) may be written as

$$\frac{\partial p_l(u)/\partial u}{p_l(u)} = \rho_l^{-1}\frac{\partial p(u)/\partial u}{p(u)}, \quad (1.16)$$

where ρ_l is land's factor share in output revenue, or $\rho_l = p_l l/(ph)$. Equation (1.16) states that the percentage change in unit rents equals the percentage change in housing prices *magnified* by the inverse of land's factor share. Since land prices alone (i.e., not capital rentals) reflect housing price changes, their percentage change will always magnify those of housing prices. Thus as we approach the city center, land rents should rise much more quickly than housing rents. The term ρ_l is usually estimated to be around 0.1 and therefore a 1% rise in housing rents should induce about a 10% rise in residential land rents.

From the information on price changes in Equation (1.14) we can demonstrate how the intensity of land use varies in a city and how population density varies. There are two measures of the intensity of land use in a city. The first is the ratio of captial to land in producing a unit of housing. We define the direct elasticity of substitution as $\sigma = d\log(k/l)/d\log(p_l/p_k)$. In the city only p_l varies with distance, so we may state, using the definition of σ and Equation (1.16),

$$\frac{\partial \log[k(u)/l(u)]}{\partial u} = \sigma\,\partial \log p_l(u)/\partial u = \frac{\sigma}{\rho_l}\partial \log p(u)/\partial u. \quad (1.17)$$

Equation (1.17) indicates that a 1% increase in housing prices as we approach the city center leads to a σ/ρ_l percent increase in the use of capital relative to land per unit of housing. Typically σ is estimated to be about 0.7 and, from above, $\rho_l = 0.1$. Therefore, a 1% change in $p(u)$ will lead to about a 7% increase in the capital-to-land ratio. This strong increase in the captial/land ratio as we approach the city center will be reflected in higher buildings. This change in the k/l ratio is illustrated in Figure 1.4a. If the rent-gradient shifts up because, say,

[13] Note that this statement is only true for infinitesimal spatial moves. For discrete spatial moves, as the price of housing changes, housing and land consumption also change. In that case, changes in housing costs holding utility constant reflect not just amenity differences but also housing consumption differences. This suggests that changes in land or housing rents can only be used to directly value amenity differences, such as the value of differential access, for infinitesimal changes in these amenities. For discrete changes one can use differences in rent expenditures to measure the value of amentity differences only if lot size is fixed.

population increases, for the same rent on capital the k/l ratio will shift up, as illustrated in Figure 1.4a. Buildings at the same location will be higher in larger cities. The increase in the k/l ratio also implies that the physical marginal product of capital is declining as we approach the CBD. Given that housing producers everywhere use capital according to the marginal productivity condition $p_k = p(u)MP_k$, this decline in MP_k matches the rise in $p(u)$, so in net the *value* of the marginal product of capital is unchanged.

The second measure of how land use intensity changes is the value of housing per unit of land. If housing production is competitive, $p(u)h(u) = p_l(u)l(u) + p_k k(u)$ or $p(u)h(u)/l(u) = p_l(u) + p_k k(u)/l(u)$. Differentiating, we get

$$\partial(p(u)h(u)/l(u))/\partial u = \partial p_l(u)/\partial u + p_k \partial[k(u)/l(u)]/\partial u.$$

Substituting in from Equations (1.14) and (1.17) we obtain

$$\frac{\partial \log[p(u)h(u)/l(u)]}{\partial u} = \left(1 + \frac{\rho_k}{\rho_l}\sigma\right)\partial \log p(u)/\partial u, \tag{1.18}$$

where $\rho_k = p_k k(u)/[p(u)h(u)]$ is capital's factor share in production revenue. Equation (1.18) states that a 1% rise in housing prices as we approach the city center will lead to a $1 + (\rho_k/\rho_l)\sigma$ percent rise in the value of housing per unit of land. For $\rho_k = 0.9$, $\rho_l = 0.1$, and $\sigma = 0.7$, a 1% rise in housing prices would lead to a 7.3% rise in the value of housing per unit of land. This is clearly a significant rise in the intensity of land use.

In terms of population density, there are two reasons why the number of people per square unit, or population density, rises as we approach the city center. First, housing consumption for equal-income people declines with higher housing prices as we approach the city center. Second, corresponding to the increasing housing prices are increasing land rents, which means, as seen in

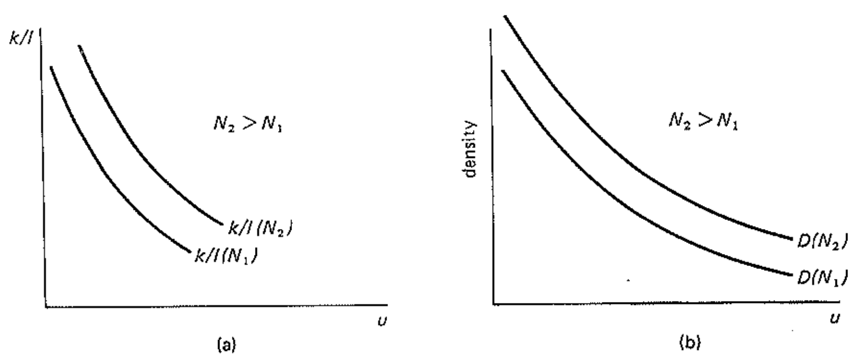

FIGURE 1.4 Capital-to-land ratios and population density.

Figure 1.4a, that less land relative to capital will be used in housing production. Both these facts indicate that per person use of land will decline as we approach the city center. Hence, population density will increase as we approach the city center. This is pictured in Figure 1.4b. Note that the density gradient will shift up if the rent gradient shifts up.

An Illustration of Land Rent Gradients

By specifying a particular production function for housing and using the results on consumption relationships from the specific utility function used in the previous section, we can illustrate a land rent gradient as well as spatial variations in density and the per person use of land. We use the production function

THIS IS CRS

$$h(u) = B' l(u)^{\alpha} k(u)^{1-\alpha}. \tag{1.11a}$$

The first-order conditions for profit maximization are ⇒ 1.19 WRONG.

$$p_l(u) = p(u)\alpha h(u)/l(u) \quad \text{and} \quad p_k = p(u)(1-\alpha)h(u)/k(u).$$

Rearranging the first condition, we get the demand for land and capital functions, or

$$l(u) = \alpha h(u)p(u)p_l(u)^{-1} \quad \text{and} \quad k(u) = (1-\alpha)h(u)p(u)p_k^{-1}.$$

Substituting in the demand equation for housing from Equation (1.2a), where $h(u) = (c/f)yp(u)^{-1}$, we can write the demand for land and capital as

$$l(u) = (\alpha c/f)yp_l(u)^{-1} \quad \text{and} \quad k(u) = (1-\alpha)(c/f)yp_k^{-1} \tag{1.19}$$

Thus the derived demand for factors is a function of income and own prices. In Equation (1.18), as we approach the CBD and land rents rise, the use of land declines and density rises. Substituting into the production function for $l(u)$ and $k(u)$ from the first-order conditions, we obtain the unit cost function

$$p(u) = Bp_l(u)^{\alpha}p_k^{(1-\alpha)}, \tag{1.12a}$$

where $B = B'\alpha^{-\alpha}(1-\alpha)^{\alpha-1}$.

The land rent gradient may be found by differentiating profits with respect to u, setting $d\pi = 0$, substituting in demand equations for $l(u)$ and $k(u)$, integrating, and then substituting in Equation (1.10) for $p(u)$. Alternatively, the land rent gradient may be found by substituting the cost function (1.12a) into the housing rent gradient (1.10) to obtain ?

$$p_l(u) = p_l(u_1)(T - tu_1)^{-d/c\alpha}(T - tu)^{d/c\alpha}, \tag{1.15a}$$

where $p_l(u_1)$ is the land rent at the city edge in agriculture. The slope of the land rent gradient is $-td/c\alpha$ and is steeper than the housing rent gradient of slope $-td/c$, as was indicated would be the case in Equation (1.16). The height of this gradient is a function of city spatial size and $p_l(u_1)$.

1.3 Aggregate Relationships in the Residential Sector

In the previous two sections, using basic consumer utility and producer profit maximization models, I examined the spatial variation in housing and land prices, housing consumption, the capital/land ratio, and density. With this information I can derive aggregate market relationships for the residential sector. These relationships are the usual ones describing aggregate demand for land and capital in the residential sector as a function of prices, income, and other variables. It is assumed that housing is a normal good with a positive income effect and a negative own-price effect; that capital and land are normal inputs with positive output effects and negative own-price effects; and that equilibrium rent gradients satisfy the four properties of market equilibrium discussed earlier. After examining general aggregate demand relationships, I illustrate these demand functions and market equilibrium using the specific functional forms for utility and production functions from Sections 1.1 and 1.2. I also use these functional forms to illustrate calculations of residential population, rents, and use of factor inputs.

The aggregate demand for residential land, given the area of the CBD, is measured by the radius of the city u_1. If we can solve for u_1 as a function of income, prices, and population, then we have solved for the urban demand for agricultural land. We start by assuming all people in the city are identical.

To solve for u_1, we calculate the residential population of the city. At each distance from the city center, the population $N(u)$ equals the total amount of land at that distance, $2\pi u_1$, divided by per person consumption of land, $l(u)$. Total population N is the sum of populations at all locations or

$$N = \int_{u_0}^{u_1} N(u)\,du = \int_{u_0}^{u_1} 2\pi u\, l(u)^{-1}\,du \qquad (1.20)$$

To derive an expression for $l(u)$, note that given the production function for housing in (1.11) and the associated profit maximization problem there is a derived demand for land where $l(u) = \tilde{l}(h(u), p_k, p_l(u))$. From Equation (1.2) we substitute in $h = h(y, p(u), p_x, p_z, e(u))$ and then we substitute for $p(u)$ and $p_l(u)$ from Equations (1.7a) and (1.15), respectively, to get $l(u) = l(y, p_x, p_z, p_k, p_l(u_1), u_1, u; t, T)$. Substituting this in (1.20) we have an equation in price and income variables as well as u that is integrated over. Thus

Equation (1.20) becomes

$$N = N(y, p_x, p_z, p_k, p_l(u_1), u_0; t, T) \qquad (1.20a)$$

Inverting this

$$u_1 = u(N, y, p_x, p_z, p_k, p_l(u_1), u_0; t, T) \qquad (1.21)$$

Equation (1.21) is in essence the city's (or at least its residential sector's) demand for agricultural land. I next argue the comparative statics properties of this function. In doing so I also investigate the comparative statics properties of residential rent gradients.

1. $\partial u_1/\partial N > 0$. If population rises, more agricultural land is demanded at the same price. This will occur if the urban land rent gradient shifts up at the initial edge of the city and the city expands farther into agriculture to take advantage of the relatively lower rents. At the new expanded city edge, land rents will again be equalized. Such a shift is depicted by the shift from R_1 to R_2 in Figure 1.3. If the rent gradient shifts up at the initial u_1 (or at any point), it must shift up at all points so as to maintain equal utility for all people. Note the alternatives to a shift up can be ruled out. A shift down with increased population would mean that the original residents with lower prices are demanding less land, contradicting the properties of the demand function for land. If the curve rotates crossing the original, relative to the initial equilibrium, some people would be worse (above the original) and others better off (below the original), which contradicts the equal utility condition.

Then to show that, as population rises, more agricultural land is demanded, it is sufficient to show that the land rent gradient must shift up as population increases. That the rent gradient shifts up can be demonstrated by showing that the contrary cannot be true. Suppose population rises and the rent gradient and spatial area of the city remain unchanged. This implies either that new people consume no land and demand no housing, or that initial residents were not maximizing utility before and are now satisfied with less land (so that new people are able to get some) at the same price. This contradicts either or both the assumptions that utility and profits are always maximized in equilibrium and that land and housing are normal economic goods. If we suppose that the rent gradient falls, the foregoing inconsistencies are even more pronounced. Therefore, the rent gradient must shift up with population and hence the city area will expand.

2. Using the same type of argument, it is possible to show the following: $\partial u_1/\partial p_l(u_1) < 0$, the normal own-price effect on factor demand; $\partial u_1/\partial y > 0$, the normal income effect on derived demand for a factor; $\partial u_1/\partial u_0 > 0$, or, ceteris paribus, the whole urban area expands if the CBD expands.

3. $\partial u_1/\partial p_k \gtreqless 0$. If the price of capital rises, it is unclear what happens to the

demand for land. While the demand for land relative to capital rises, the relative consumer demand for housing and both factors falls, since housing is now more costly to produce and purchase.

4. $\partial u_1/\partial t < 0$. This indicates that increased cost of access to the CBD causes people to crowd closer to the CBD. If t rises, people farthest from the CBD, say at u_1, have a greater absolute increase in commuting time than people nearer the CBD because they travel greater distances. Therefore people nearer the CBD relative to those farther away must experience an increase in land rents to offset their increased relative access advantage. This maintains a stable spatial equilibrium between those near the CBD and those farther away. This upward rotation in the rent gradient will reduce demand for land everywhere.[14] Hence u_1 will decline.

The urban demand curve for argricultural land is illustrated in Figure 1.5. The supply curve of agricultural land is also pictured and an equilibrium illustrated. Although the supply of land is drawn as infinitely elastic at $p_l(u_1)$, it could be upward sloping. For example, if the city buys agricultural produce from its hinterland and there are costs to the farmers of shipping produce from the city market, then agricultural land rents will vary with relative market access or distance to the city center. In certain circumstances, if the radius of agricultural production rises with city size, so will the level of agricultural land rents.[15] Then the opportunity cost of land to the city will rise.

The aggregate demand for capital can be specified in the same fashion and shown to have regular properties. The analysis can be expanded to include

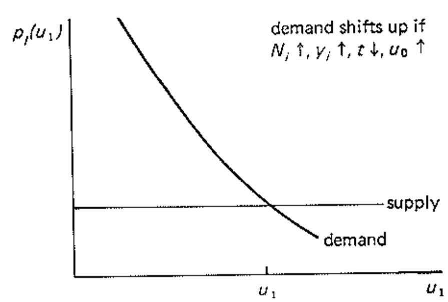

FIGURE 1.5 Urban demand for agricultural land.

[14] To actually rule out the alternative of an anticlockwise rotation with u_1 shifting out, it is sufficient to assume utility is separable in leisure, so housing demand depends only on income and prices.

[15] As the city expands, if in the agricultural area the *difference* in access between the farmers closest and those farthest from the CBD increases, then agricultural rents should increase.

different income groups, where we would then hypothesize a demand for agricultural land function.

$$u_1 = \tilde{u}(p_l(u_1), N_1, N_2, N_3, \ldots, y_1, y_2, y_3, \ldots, u_0, p_k, p_x, p_z; t, T),$$

where N_i is the number of people earning income y_i. The comparative static properties of this function are argued in the same way as when there is only one type of person. For example, in arguing $\partial u_1/\partial N > 0$, I use the same method, except now I must also show if the rent gradient rises in one income segment, it rises in all segments. If land rents rise in one segment, either they rise in other segments to equalize land rents at the border of competing uses or the area of this segment will expand into other segments, reducing their size. This reduction in size in itself will drive up prices in these other segments to equate demand with the reduced supply.

Specific Functional Forms

These aggregative relationships are illustrated using the logarithmic linear production and utility functions specified earlier. These illustrations will be used later in presenting an aggregative model of a city. For the illustration it is assumed that *all* consumers have *identical tastes and incomes.* To illustrate aggregate relationships is very complicated when incomes and/or tastes vary (see Montesano, 1972). The basic problem is in integrating over space when there are different types of consumers whose location is endogenous. Since the objective here is to illustrate an aggregative model, I choose to simplify and assume that consumers are identical.

We start by calculating the residential population of a city. At each distance from the city center the population $N(u)$ equals the total amount of land at that location, $2\pi u$, divided by per person consumption of land. Total population N is the sum of populations at all locations or

$$N = \int_{u_0}^{u_1} N(u)\,du = \int_{u_0}^{u_1} 2\pi u l(u)^{-1}\,du.$$

Using our logarithmic linear utility and production functions, we substitute into this equation for $l(u)$ from the demand-for-land equation (1.20) where $l(u) = (\alpha c/f)yp_l(u)^{-1}$, and then we substitute for $p_l(u)$ from the rent gradient equation (1.19) where $p_l(u) = p_l(u_1)(T - tu_1)^{-d/c\alpha}(T - tu)^{d/c\alpha}$. This yields

$$N = \int_{u_0}^{u_1} 2\pi u(\alpha c/fy)^{-1} p_l(u_1)(T - tu_1)^{-d/c\alpha}(T - tu)^{d/c\alpha}\,du.$$

Integrating, we get

$$N = C_1 p_l(u_1) y^{-1} t^{-2} s(t, u_0, u_1) \tag{1.22}$$

where

$$C_1 = 2\pi\left(\frac{f}{c\alpha}\right)\left(\frac{d}{c\alpha}+2\right)^{-1}\left(\frac{d}{c\alpha}+1\right)^{-1}$$

and

$$s(t,u_0,u_1) = (T-tu_1)^{-d/c\alpha}(T-tu_0)^{1+d/c\alpha}\left[T+tu_0\left(\frac{d}{c\alpha}+1\right)\right]$$
$$-(T-tu_1)\left[T+tu_1\left(\frac{d}{c\alpha}+1\right)\right] > 0.^{16}$$

An inverse of Equation (1.22) is an equation for u_1, the urban demand for agricultural land, where $u_1 = u(p_l(u_1), N, y, t, u_0, p_k)$. The properties of this function may be illustrated by differentiating (1.22) and rearranging terms to get

$$e_1\,du_1 = (dN/N) + (dy/y) - [dp_l(u_1)/p_l(u_1)] - e_2\,dt + e_3\,du_0 \quad (1.23)$$

where $e_1, e_2, e_3 > 0$.[17] Equation (1.23) illustrates the properties of the urban demand function for agricultural land that were discussed earlier. There is the own-price relationship, $\partial u_1/\partial p_l(u_1) < 0$. There are the aggregate demand relationships for population and income, $\partial u_1/\partial N$, $\partial u_1/\partial y < 0$. Finally, $\partial u_1/\partial t < 0$ and $\partial u_1/\partial u_0 > 0$.

The aggregate demand for capital is the demand for capital per house summed over all households. The amount of capital per house is $k(u)$ and the population at each distance from the CBD is $2\pi ul(u)^{-1}$. Therefore at each radius the demand for capital is $k(u)2\pi ul(u)^{-1}$. From Equation (1.19) we can substitute into this relationship $k(u) = (1-\alpha)c/fyp^{-1}$ and $l(u) = \alpha c/fyp_l(u)^{-1}$. Therefore the aggregate demand for capital is[18]

$$K = \int_{u_1}^{u_0} 2\pi uk(u)/l(u)\,du = \int_{u_1}^{u_0} 2\pi u\alpha/(1-\alpha)p_k^{-1}p_l(u)\,du.$$

To evaluate this, we can substitute in Equation (1.20) for $p_l(u)$ and integrate. Alternatively, we can employ other information to solve for K. Given logarithmic linear utility functions, consumers spend a fixed fraction of their income on housing, or from (1.2a), $p(u)h(u) = (c/f)y$. From equation (1.19) we also know that the share of any factor is a fixed proportion of housing costs. Specifically, $p_k k(u) = (1-\alpha)p(u)h(u)$. Combining these relationships, we see that each consumer buys $k(u) = (1-\alpha)(c/f)yp_k^{-1}$. This expression contains no spatial variables and, therefore, aggregate demand is simply

$$K = (1-\alpha)(c/f)yp_k^{-1}N. \quad (1.24)$$

As before, this aggregate demand function has the normal properties that $\partial K/\partial p_k < 0$ for the own-price effect and $\partial K/\partial y, \partial K/\partial N > 0$ for the income and population effects.

Finally, note that total rents at each location are the unit rent $p_l(u)$ multiplied by the amount of land at each location $2\pi u$. Therefore total residential rents are

$$\text{Rents}_{\text{res}} = \int_{u_0}^{u_1} 2\pi up_l(u)\,du.$$

We can substitute in Equation (1.20) for $p_l(u)$ and integrate. Alternatively note that per person land rents are a fixed proportion α of housing costs, which in turn are a fixed proportion of income c/f. Therefore, per person rents are $(\alpha c/f)y$, and total residential rents are

$$\text{Rents}_{\text{res}} = (\alpha c/f)yN. \quad (1.25)$$

16 Clearly, for $N > 0$, $s > 0$. If $u_0 = u_1$, $s = N = 0$, since there is no residential area. For $u_1 > u_0$ we assume that parametric values are such that $s > 0$. For any reasonable parametric values, $s > 0$.

17
$$e_1 = \frac{td}{c\alpha}s(t,u_0,u_1)^{-1}\left\{(T-tu_1)^{-(d/c\alpha)-1}(T-tu_0)^{(d/c\alpha)+1}\left[T+tu_0\left(\frac{d}{c\alpha}+1\right)\right]-T + 2tu_1\left(1+\frac{c\alpha}{d}\right)\right\} > 0,$$
$$e_2 = e_1t^{-1}u_1 - 2t^{-1} - t^{-1}u_0e_3 > 0,$$
$$e_3 = u_0t^2s(t,u_0,u_1)^{-1}(T-tu_1)^{-(d/c\alpha)}(T-tu_0)^{(d/c\alpha)}\left(\frac{d}{c\alpha}+1\right)\left(\frac{d}{c\alpha}+2\right) > 0.$$

These variables are unambiguously positive, except for e_2, providing $s > 0$. As for s, $e_2 > 0$ for reasonable parametric values. For future reference, note that it is possible to show that $(T-tu_1)e_1c\alpha/(td) > 1$ or that $(T-tu_1)$ multiplied by the bracketed part of e_1 and $s(t,u_0,u_1)^{-1}$ is greater than one. This latter expression reduces to

$$\frac{g(u_1) + (T-tu_1)2tu_1(1+c\alpha/d)}{g(u_1) - (T-tu_1)tu_1(d/c\alpha+1)},$$

where $g(u_1)$ is some positive expression and both numerator and denominator are positive. As long as $T > tu_1$ or leisure is positive, this expression is greater than one.

18 Alternatively the aggregate demand for capital equals the demand for capital per unit of land summed over all units of land. From footnote 12 the amount of capital and land per unit of housing are, respectively, $\partial p/\partial p_k$ and $\partial p/\partial p_l$. Therefore $K = \int_{u_0}^{u_1} 2\pi u(\partial p/\partial p_k)/[\partial p/\partial p_l(u)]\,du$. To solve this we substitute in from Equation (1.12a), which then gives us the equation in the text.

2. THE BUSINESS SECTOR

2.1 Firms and Spatial Equilibrium in the Central Business District

Firms in the CBD produce the city's traded, or export good. As noted earlier, for cities to exist there must be some type of scale effects in production. I use a simple specification consistent with maintaining perfect competition (Chipman, 1970). Second, to have a CBD with centrally located production, in this chapter we assume firms want access to a transport-retailing node at the center of the city, from which to export or sell their products locally. In return for its exports the city imports z from other places, through the transport node.

The firm production function is

$$x(u) = G(N)x(k(u), n(u), l(u)). \tag{1.26}$$

A Hicks neutral shift factor $G(N)$ indicates economies of scale that are dependent on city employment in x activity (the only employment source in the city); $\partial G/\partial N \geq 0$. These scale economies are the basis for agglomeration of population in the city. They are at the industry level and may be experienced by any entering firm. Each firm behaves as though $G(N)$ were exogenous. The x function denotes the firm's own technology, where output, given $G(N)$, is a function of capital $k(u)$, labor $n(u)$, and land $l(u)$. The x function is linear homogeneous and hence $G(N)x$ is homothetic.

In the shipping of products to the marketing–transport node at the city center, transport services are produced with units of x, which corresponds to the evaporation formulation of transport costs in international trade. It costs firms t_x of a unit of x to ship one unit of x one unit distance. The quantity of x actually sold in the city center is $x(u)(1 - t_x u)$ or the revenue received for each unit of x produced is $p_x(1 - t_x u)$.

These assumptions about production have a number of implications. First, in maximizing profits, the firm pays factors the value of their perceived marginal product or, for wages, $p_n = p_x(1 - t_x u)G(N)\partial x/\partial n$. Total factor payments then are $p_x(1 - t_x u)G(N)[(\partial x/\partial n)n + (\partial x/\partial k)k + (\partial x/\partial l)l]$, which, from Euler's theorem, equals $p_x(1 - t_x u)G(N)x$ if the x function is linear homogeneous. Therefore firm factor payments exhaust firm revenue. This fact plus the fact that $G(N)$ is an external scale effect that affects all firms equally ensures that perfect competition is stable and feasible.

Second, production technology may be alternatively described by the unit cost function $p(N, p_n, p_k, p_l(u))$. This function is nonincreasing in N, increasing and linear homogeneous in prices, and subsumes efficient usage of factor inputs. With perfect competition, the firm's net price should equal unit

costs or[19]

$$p_x(1 - t_x u) = p(N, p_n, p_k, p_l(u)). \tag{1.27}$$

Finally, as long as $G(N)$ increases with N, for the same factor ratios the marginal products of factors increase continuously. Moreover, because $G(N)$ is a Hicks neutral shifter, the marginal products of factors increase by the same proportion for the same factor ratios. If, on the other hand, the scale effect were relatively labor saving compared to capital, this would imply that the marginal product of capital would be more beneficially affected than the marginal product of labor as a city grows. Then, for example, either capital rentals would rise relative to wages if city factor supplies are fixed (and factor prices variable) or the city's relative demand for labor would fall if factor prices are fixed (and factor supplies are variable). Clearly, assuming $G(N)$ is a Hicks neutral or nonneutral shift factor has important implications for the relative use of capital and labor in various size cities. Normally, scale efficiencies are assumed to be neutral in aggregate.

A firms spatial equilibrium is described as follows. Firm profits are $\pi = p_x(1 - t_x u)x(u) - p_l(u)l(u) - p_k k(u) - p_n n(u)$. At the firm's profit-maximizing location, $\partial \pi/\partial u = 0$ or

$$l(u)\partial p_l(u)/\partial u = -x(u)p_x t_x < 0. \tag{1.28}$$

Unit land rents decline with distance from the city center, as transport costs increase. The change in total rents exactly equals the increase in transport costs $[x(u)p_x t_x]$ of moving a unit distance. Therefore firm profits are unchanged and the firm does not benefit by moving.

To find the slope and height of the commercial land rent gradient, we first note that if all producers are identical in technology and ability, profits from producing x must be everywhere equal. If the x industry is competitive, profits will be everywhere zero and net price will always equal unit production costs. Therefore, to find the slope of the rent gradient, we can differentiate the unit cost function and do appropriate substitutions, or we can differentiate the profit function and set $d\pi = 0$.[20] Alternatively we observe that, since Equation (1.28) specifies a relationship that must hold for individual producers to be in equilibrium, it also gives the slope of the equilibrium rent gradient. The height of the rent gradient at any u can be solved directly from Equation (1.27), given p_x, t_x, and other variables.

[19] See footnote 11 on the existence of unit cost functions.

[20] Differentiating the unit cost function gives us $-p_x t_x du = \partial p/\partial p_l(u)\, \partial p_l(u)/\partial u\, du$. From Shephard's lemma (footnote 13) $l(u)/x(u) = \partial p/\partial p_l(u)$. Substituting this in yields Equation (1.28).

In long-run equilibrium in the urban land market, land rents in the residential sector at u_0, the boundary of the CBD, equal land rents at u_0 in the business sector, or

$$p_l(u_0)_{\text{res}} = p_l(u_0)_{\text{bus}}. \tag{1.29}$$

For this equilibrium to be stable, as indicated in the discussion of residential equilibrium with different types of consumers (p. 12), businesses must be able to outbid residences for land interior to u_0. (This is the stability condition implicit in the rent gradients in Figure 1.2.) Therefore, the slope of the bid rent curve for businesses must exceed that for residences at u_0, or

$$|\partial p_l(u_0)_{\text{res}}/\partial u| \leq |\partial p_l(u_0)_{\text{bus}}/\partial u|,$$

or, using (1.14) and (1.28), with one firm at u_0

$$N(u_0)tp_e(u) \leq x(u_0)p_x t_x. \tag{1.30}$$

Specific Functional Forms

The foregoing points are illustrated with a logarithmic linear production function of the form

$$x(u) = G(N)C'l(u)^\gamma k(u)^\beta n(u)^\delta, \qquad \gamma + \beta + \delta = 1. \tag{1.26a}$$

From the first-order conditions for profit maximization, we get the marginal productivity conditions, which may be rewritten as factor demand equations, where, for $\tilde{p}_x = p_x(1 - t_x u)$,

$$l(u) = \gamma\tilde{p}_x x(u)/p_l(u), \qquad n(u) = \delta\tilde{p}_x x(u)/p_n, \qquad k(u) = \beta\tilde{p}_x x(u)/p_k. \tag{1.31}$$

The unit cost function corresponding to this production function is obtained by substituting into the production function for $l(u)$, $k(u)$, and $n(u)$ from Equation (1.31) to get

$$p_x(1 - t_x u) = G(N)^{-1}Cp_n{}^\delta p_k{}^\beta p_l(u)^\gamma, \tag{1.27a}$$

where $C = (C')^{-1}\delta^{-\delta}\beta^{-\beta}\gamma^{-\gamma}$. Rearranging Equation (1.27a) yields a rent gradient of

$$p_l(u) = p_x^{1/\gamma}(1 - t_x u)^{1/\gamma}C^{-1/\gamma}G(N)^{1/\gamma}p_n^{-\delta/\gamma}p_k^{-\beta/\gamma}. \tag{1.32}$$

If we employ the condition that $p_l(u_0)_{\text{res}} = p_l(u_0)_{\text{bus}}$, we may write business rents as a function of $p_l(u_0)$ and distance from the city center. Substituting u_0 for u in (1.32), dividing the result by (1.32), and rearranging terms yields

$$p_l(u) = p_l(u_0)(1 - t_x u_0)^{-1/\gamma}(1 - t_x u)^{1/\gamma}. \tag{1.33}$$

The height of the business rent gradient is determined by the rent at the edge of the residential sector $p_l(u_0)$, the size of the CBD u_0, and transport costs t_x.

2.2 Aggregate Relationships in the Central Business District

Having examined equilibrium conditions for the individual producer and in the general land market, we can now determine aggregate employment of capital and labor, total output, and income from the rent of land in the CBD. Because of scale economies, general aggregate relationships for factor inputs cannot be proved by contradiction proofs. Therefore, we proceed directly to illustrating the problem with a logarithmic linear production function.

At each location from the city center the aggregate employment of labor is the employment of labor per unit of land, $n(u)/l(u)$, summed over all units of land, $2\pi u$. From the factor demand Equations (1.31), $n(u)/l(u) = \delta/\alpha p_l(u)/p_n$. Therefore total employment in the CBD is[21]

$$N = \int_0^{u_0} 2\pi u\left(\frac{\delta}{\gamma}\right)\left(\frac{p_l(u)}{p_n}\right)dN.$$

Substituting in (1.32) for $p_l(u)$ and integrating, we get

$$N = C_2 p_x^{1/\gamma}G(N)^{1/\gamma}p_n^{-1-\delta/\gamma}p_k^{-\beta/\gamma}t_x^{-2}f(t_x, u_0), \tag{1.34}$$

where

$$C_2 = C^{1/\gamma}2\pi\left(\frac{\delta}{\gamma}\right)\left(\frac{1}{\gamma} + 2\right)^{-1}\left(\frac{1}{\gamma} + 1\right)^{-1}$$

and

$$f(t_x, u_0) = \left\{1 - (1 - t_x u_0)^{1/\gamma + 1}\left[1 + t_x u_0\left(\frac{1}{\gamma} + 1\right)\right]\right\}.$$

Equation (1.34) describes total employment of labor in the x industry. It is sometimes interpreted as an aggregate demand function for labor in the x industry. Interpreting it as a demand function must be done with care since it

[21] Note that from the formulation of the production function, firm size is indeterminate, so that production activity is summed over locations rather than firms. The aggregate demand for labor may also be derived as follows. From footnote 20 the demand for labor per unit of output is $\partial p/\partial p_n$ and the unit demand for land is $\partial p/\partial p_l(u)$. Hence total employment at location u is $2\pi u(\partial p/\partial p_n)/[\partial p/\partial p_l(u)]$. Evaluating these derivatives using the unit cost function in Equation (1.27a) yields the expression above for N.

characterizes neither the demand of individual producers nor the city's demand for labor (even though the x industry is the only employer in the city). As we shall see in Chapter 2, to find the *city's* demand for labor, we have to incorporate information from the residential sector of the city to obtain a city demand function for population.

To examine the x industry's employment of labor, we differentiate (1.34) to obtain

$$\frac{dN}{N} = \frac{1}{\gamma - \varepsilon}\frac{dp_x}{p_x} - \frac{\gamma + \delta}{\gamma - \varepsilon}\frac{dp_n}{p_n} - \frac{\beta}{\gamma - \varepsilon}\frac{dp_k}{p_k} + \frac{e_4}{1 - \varepsilon/\gamma}du_0 - \frac{e_5}{1 - \varepsilon/\gamma}dt_x \tag{1.35}$$

where $\varepsilon = [dG(N)/dN]N/G(N)$ is the elasticity of the scale economy shift factor with respect to N and $e_4 > 0$, $e_5 \lesseqgtr 0$.[22] A term we shall use frequently in the next chapter, ε indicates the extent of scale economies at the *margin* of additional employment. In general, we shall assume that ε is declining with city size, or $\partial\varepsilon/\partial N < 0$, indicating that scale economies are larger at the margin when cities are small.

In examining Equation (1.34), we see that if $\varepsilon < \gamma$ or ε is relatively small, this equation will possess normally expected properties. An increase in employment is associated with a rise in output price, a decline in own input price, an increase in CBD area, and in some cases a decline in transport costs.[23] A rise in p_k, for p_x fixed, is associated with a decline in both capital and labor employment, although labor employment can be shown to increase relative to capital employment [compare the dp_k/p_k coefficients in Equations (1.35) and (1.37) for $\gamma > \varepsilon$].

If marginal scale effects are large, such that $\varepsilon > \gamma$, Equation (1.35) has seemingly unusual properties. For example, a rise in wages is associated with an increase in labor employed. However, if one interprets Equation (1.35) as stating that when scale effects are large, wages can rise as employment increases (for the same size CBD), then that makes sense.

At each location from the city center the aggregate employment of capital is $k(u)$, which can also be stated as the employment of capital per unit

22
$$e_4 = f(t_x, u_0)^{-1}(1 - t_x u_0)^{1/\gamma}\left(\frac{1}{\gamma} + 1\right)\left(\frac{1}{\gamma} + 2\right)t_x^2 u_0 > 0,$$
$$e_5 = 2t_x - t_x^{-1}u_0 e_4 > 0.$$

23 For the same u_0, if t_x increases, rents should rise nearer the city center relative to farther away due to increased premiums on access to the transport node. This increase in rents will lead to a greater use of N relative to land but a potentially offsetting reduction in demand for all factors.

of land, $k(u)/l(u)$, summed over all units of land $2\pi u$. From the factor demand Equation (1.31), $k(u)/l(u) = \beta/\gamma)p_l(u)/p_k$. Therefore total employment in the CBD is

$$K = \int_0^{u_0} k(u)\,du = \int_0^{u_0} 2\pi u k(u)/l(u)\,du = \int_0^{u_0} 2\pi u(\beta/\gamma)p_l(u)/p_k\,du.$$

To evaluate this we can substitute in Equation (1.32) for $p_l(u)$ and integrate. Alternatively, from Equation (1.31) two first-order conditions are $p_n = \delta\tilde{p}_x x(u)/n(u)$ and $p_k = \beta\tilde{p}_x x(u)/k(u)$. Combining and solving out $p_x x$, we get $k(u) = (\beta/\delta)p_k^{-1}p_n n(u)$. Since $N = \int_0^{u_0} n(u)\,du$, we may then state

$$K_{\text{bus}} = \int_0^{u_0} k(u)\,du = (\beta/\delta)p_n p_k^{-1} N. \tag{1.36}$$

Differentiating (1.36) yields the properties of the employment function for capital.

$$\frac{dK}{K} = \frac{dp_n}{p_n} - \frac{dp_k}{p_k} + \frac{dN}{N}.$$

To make this comparable with the function for labor, we substitute in Equation (1.35) for dN/N to get

$$\frac{dK}{K} = \frac{1}{\gamma - \varepsilon}\frac{dp_x}{p_x} - \frac{\delta + \varepsilon}{\gamma - \varepsilon}\frac{dp_n}{p_n} - \frac{\beta + \gamma - \varepsilon}{\gamma - \varepsilon}\frac{dp_k}{p_k} + \frac{e_4}{1 - \varepsilon/\gamma}du_0 - \frac{e_5}{1 - \varepsilon/\gamma}dt_x. \tag{1.37}$$

When scale effects are small, the properties of this employment of capital equation will be similar to those of the properties of the labor equation. Increases in capital usage are associated with increases in output price and CBD area and declines in own price. An increase in the price of labor is associated with an absolute decline in capital usage, but a rise relative to labor usage [compare the coefficients of dp_n/p_n in (1.35) and (1.37) if $\gamma > \varepsilon$]. As before, if scale effects are large, the properties of Equation (1.37) may seem unusual but are plausible.

For future reference we define expressions for total CBD rents and total output actually retailed at the transport–marketing node.

$$\text{Rents}_{\text{bus}} = \int_0^{u_0} 2\pi u p_l(u)\,du,$$

$$X = \int_0^{u_0} x(u)(1 - t_x u)\,du$$

To solve these equations we can do appropriate substitutions and integrate. Alternatively, we note from the labor marginal productivity condition that $x(u) = p_n n(u)\,\delta^{-1}\tilde{p}_x^{-1}$ and therefore using Equation (1.34), where

$$N = \int_0^{u_0} n(u)\,du = \int_0^{u_0} 2\pi u\,\delta/\gamma p_l(u)/p_n\,du,$$

it is possible to evaluate both equations directly in terms of labor employment. Doing this, we find

$$\text{Rents}_{\text{bus}} = (\gamma/\delta)p_n N, \tag{1.38}$$

$$X = \delta^{-1}p_x^{-1}p_n N. \tag{1.39}$$

These expressions for CBD rents and output plus the expressions for CBD employment of capital and labor define the primary aggregate relationships in the CBD.

2

An Aggregative Model of a Simple City

In this chapter an aggregative model of a city is presented and analyzed. The purpose of developing an aggregative model is to solve for the city's total demand for population, factor incomes, and equilibrium city size, and to show how various economic characteristics of the city vary with city size. Given this analysis, we can solve for equilibrium city size by postulating a supply function of people to the city and then analyzing the behavior of economic agents in limiting city sizes. In addition to being used to solve for equilibrium city size, the model that is developed can be used to do comparative static analyses of the long-run effect on city size and other economic characteristics of changes in commuting costs, property taxes, and other variables.

In the previous chapter I developed a model of the residential and business sectors of a city and derived functions describing aggregate demands for labor, capital, and land in those sectors. The aggregative model combines these two sectors to find the city's total demand for factors, factor income, and city size given the supply functions of factors available to the city. The model is solved using the specific functional forms introduced in Chapter 1. In doing this it is assumed that city residents have identical incomes and tastes. In a sense a partial equilibrium framework is assumed since the city is treated as a small entity relative to the rest of the economy and world. As such, the city borrows capital at a fixed rental rate in national or international markets; it buys and sells traded goods at fixed prices or at least faces a given demand function for its exports; and it has an exogenous supply function of labor.