1

MAX $V(c, q)$ [CONSUMER PROBLEM]

S.T. $y = c + pq + tx$

F.O.C $pV_c = V_q$ (1)

FREE-MOBILITY $V(y - pq - tx, q) = \bar{u}$ (2)

(2) $\Rightarrow$ $V_c[-pq_t - p_t q - x] + V_q q_t = 0$ (3)

(2) $\rightarrow$ (3) $\Rightarrow$ $p_t = -x/q < 0$ FOR $x > 0$. (4)

$\max_s \; l[ph(s) - r - iS]$ [HOUSING FIRM PROBLEM]

F.O.C $\Rightarrow$ $ph_s = i$ (5)

FREE-ENTRY $\Rightarrow$ $ph(s) - r - iS = 0$ (6)

(6) $\Rightarrow$ $p_t h + ph' \cdot S_t - r_t - iS_t = 0$

USING (5) $\Rightarrow$ $r_t = h \cdot p_t$

$\boxed{\therefore \text{ USING (4) WE HAVE } r_t = \dfrac{-xh}{q} < 0 \text{ FOR } x > 0}$

$D \equiv hq^{-1}$ SO WE HAVE:

$$D_t = \frac{-h(s) q_t + h' S_t q}{q^2} \quad (7)$$

$h, h', q, q^2 > 0$ BY ASSUMPTION. THAT JUST LEAVES q_t, S_t.

BUT $q_t > 0$ B/C $p_t < 0$ AND q_t IS A COMPENSATED DEMAND.

$\longrightarrow$

TOTAL DIFFERENTIATING (5), WE HAVE

$$p_t h' + p h'' \cdot S_t = 0$$

$$\Rightarrow \quad S_t = \frac{-p_t h'}{p h''} = -\left[-\frac{x}{f} \frac{h'}{p h''} \right] < 0 \quad (8)$$

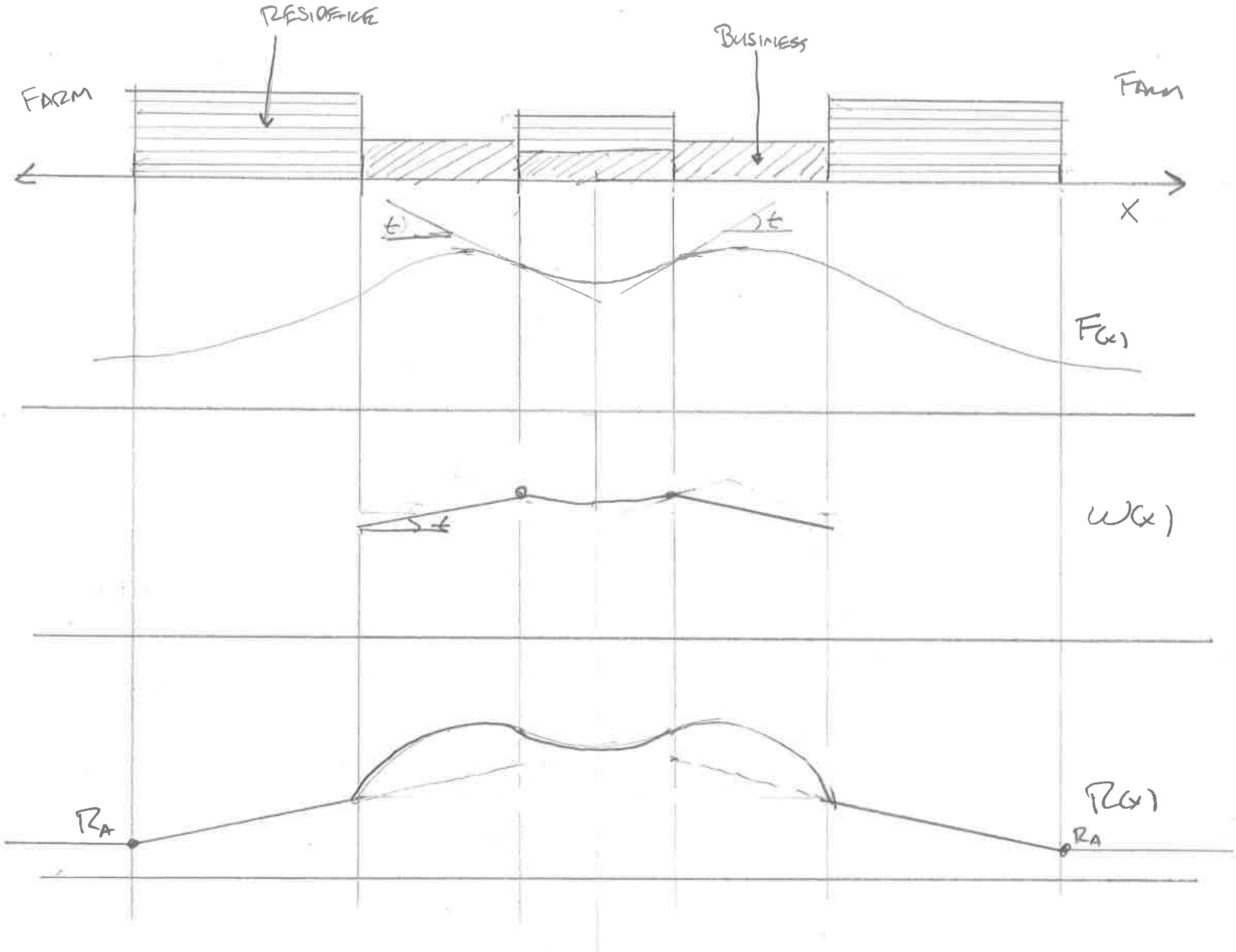
USING (8) IN (7), TOGETHER WITH $f_t > 0$, WE HAVE $D_t < 0$

THAT IS, AS $t \uparrow$, AT ANY x, WE HAVE LAND PRICES AND DENSITY $\downarrow$.

INTUITIVELY, AS $t \uparrow$ COMMUTE COSTS GO UP AT EACH x. AS COMMUTE COST $\uparrow$, LAND PRICES $\downarrow$ TO PRESERVE CONSTANT u. BUT THIS MEANS (1) THE CAPITAL LAND RATIO SHOULD FALL (2) HOUSING PER PERSON $\uparrow$. TOGETHER THIS MEANS $D \downarrow$.

ANOTHER WAY TO THINK ABOUT THIS IS, AS $t \uparrow$ WE ARE RESCALING THE x AXIS, AND EACH x "LOOKS LIKE" A LOCATION THAT WAS MORE REMOTE WITH SMALLER t.

FUJITA + OGAWA PARTIALLY MIXED EQUIL



1. CENTER IS MIXED $\Rightarrow$ WORK WHERE YOU LIVE
   $\Rightarrow$ SLOPE OF WAGE GRADIENT LESS THAN t.
2. PEOPLE IN ADJACENT BUSINESS DISTRICTS COMMUTE FROM OUTSIDE $\Rightarrow$ WAGE GRADIENT HAS SLOPE t AND RENT GRADIENT ALWAYS ABOVE tx.

(1)

- $s \in [s_1, s_2]$ ~ AMENITY
  x ~ NUMERAIRE CONSUMPTION GOOD
  l^c ~ RESIDENTIAL LAND
  k ~ UTILITY LEVEL
  w, I ~ WAGE, NON-WAGE INCOME
  r ~ LAND RENT

N ~ LABOR = PEOPLE
l^p ~ PRODUCTION LAND
$X = f(N, l^p; s)$ ~ CRS.

(1) $L = l^c + l^p$

- CONSUMERS SOLVE: $\max u(x, l^c; s)$ s.t. $x + r l^c = w + I$ $\Rightarrow$ (2) $V(w, r; s) = k$ (FREE MOBILITY)

- FIRMS ~~HAVE~~ UNIT COST FUNCTION

$$(3) \quad c(w, r; s) = 1$$

(FREE ENTRY + $\pi = 0$ + $P_X \equiv 1$.)

WITH CRS $\Rightarrow$ (4) $c_r = \frac{l^p}{X}$, (5) $c_w = N/X$

AN EQUILIBRIUM MUST SATISFY (1)-(5).

TOTAL DIF (2) + (3) $\Rightarrow$

$$V_w \frac{dw}{ds} + V_r \frac{dr}{ds} = +V_s \quad (6)$$

$$c_w \frac{dw}{ds} + c_r \frac{dr}{ds} = -c_s \quad (7)$$

SOLVE (6)+(7) FOR

$$\frac{dr}{ds} = \frac{-V_w c_s + V_s c_w}{V_w c_r - V_r c_w} \equiv \frac{V_w c_s - V_s c_w}{\Delta}$$

SOLVE (6)+(7) FOR

$$\frac{dw}{ds} = \frac{-V_s c_r + V_r c_s}{V_w c_r - V_r c_w} \equiv \frac{V_s c_r - V_r c_s}{\Delta}$$

2

USING (4) + (5) WE HAVE

$$\Delta = -V_r C_w + V_w C_r = -V_r \frac{N}{X} + V_w \frac{l^p}{X}$$

$$= V_w \left[-\frac{V_r}{V_w} \frac{N}{X} + \frac{l^p}{X} \right]$$

USING ROY'S IDENTITY,

$$= V_w \left[+l^c \frac{N}{X} + \frac{l^p}{X} \right]$$

$$= +V_w \left[\frac{l^c N + l^p}{X} \right]$$

$$= \frac{+V_w L}{X}$$

FROM USING CRS OF $C(\cdot)$ IN (7)

$$\Rightarrow -C_s = \frac{N}{X} \frac{dw}{ds} + \frac{l^p}{X} \frac{dr}{ds}$$

$$\Rightarrow C_s = -\left[\frac{N}{X} \frac{dw}{ds} + \frac{l^p}{X} \frac{dr}{ds} \right] \quad (8)$$

USING CRS OF $C(\cdot)$ IN (6)

$$\Rightarrow \frac{dw}{ds} + \frac{V_r}{V_w} \frac{dr}{ds} = \frac{V_s}{V_w}$$

(USE ROY'S IDENTITY) $\left[\frac{V_r}{V_w} = -l^c \right]$

$$\Rightarrow \frac{dw}{ds} = l^c \frac{dr}{ds} + \frac{V_s}{V_w}$$

$$\Rightarrow \frac{dw}{ds} = l^c \left[\frac{-V_w C_s + V_s C_w}{\Delta} \right] + \frac{V_s}{V_w}$$

$$\Rightarrow \frac{dw}{ds} = \frac{+X l^c}{V_w L} \left[-V_w C_s + V_s \frac{N}{X} \right] + \frac{V_s}{V_w}$$

$\rightarrow$

(8)

$$\Rightarrow \frac{dw}{ds} = \frac{-x\ell^c}{L} \cdot c_s - \frac{N\ell^c}{L} \cdot \frac{V_s}{V_w} + \frac{V_s}{V_w}$$

$$P_s^* \equiv V_s/V_w \Longrightarrow \frac{dw}{ds} = \frac{-x\ell^c}{L} c_s - \frac{N\ell^c}{L} P_s^* + P_s^*$$

$$\frac{dw}{ds} = -\frac{x\ell^c}{L} c_s + \frac{L - N\ell^c}{L} P_s^*$$

$$\frac{dw}{ds} = \frac{-x\ell^c}{L} c_s + \frac{\ell^P}{L} P_s^* \qquad (9)$$

USING (8) TO SUBSTITUTE FOR c_s IN (9)

$$\frac{dw}{ds} = +\frac{x\ell^c}{L}\left[c_w \frac{dw}{ds} + c_r \frac{dr}{ds}\right] + \frac{\ell^P}{L} P_s^*$$

USING CRS

$$= +\frac{x\ell^c}{L}\left[\frac{N}{x}\frac{dw}{ds} + \frac{\ell^P}{x}\frac{dr}{ds}\right] + \frac{\ell^P}{L} P_s^*$$

$$\frac{\ell^P}{L} P_s^* = \frac{dw}{ds} - \frac{x\ell^c}{L}\left[\frac{N}{x}\frac{dw}{ds} + \frac{\ell^P}{x}\frac{dr}{ds}\right]$$

$$\frac{\ell^P}{L} P_s^* = \frac{dw}{ds} - \frac{\ell^c N}{L}\frac{dw}{ds} - \frac{\ell^c \ell^P}{L}\frac{dr}{ds}$$

$$P_s^* = \left(\frac{L}{\ell^P} + \frac{\ell^c N}{\ell^P}\right)\frac{dw}{ds} - \ell^c \frac{dr}{ds}$$

$$\Longrightarrow \boxed{P_s^* = \frac{dw}{ds} - \ell^c \frac{dr}{ds}}$$