EC1410 – Spring 2025

Midterm
8:40-9:40am, March 10, 2025
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You will have 60 minutes to complete this exam. No notes or books are allowed, but you may use a calculator. Cell phones and any device with a wireless connection must be off. Anyone still working on their exam after time is called will be subject to an automatic 10 point penalty.

When you write up your answers, your goal should be to (1) be correct, and (2) convince your reader that your answer is correct. Answers which do not achieve these goals will not be awarded full credit. To accomplish the second goal, it is helpful if your work is legible and if all steps are presented, possibly with a line of explanation. Total points =100. Points assigned to each problem are indicated in parentheses.

This exam has TWO pages.

1. Consider the monocentric city model with housing and let $\overline{u} = 3$.

   (a) (10) Let the household's problem be given by:

   $$\max_{c,h,x} c^{1/2}h^{1/2}$$
   $$\text{s.t. } w = c + ph + 2tx$$

   Let $\widetilde{w} = w - 2tx$. Use the first-order condition of the household's problem with respect to h to find h^* in terms of p and $\widetilde{w}$.

   (b) (5) Solve for p^* in terms of $\widetilde{w}$.

   (c) (5) Solve for h^* in terms of w, t, and x.

   (d) (5) The technology for producing housing is constant returns to scale and can be written as $h_s(S) = S^{2/3}$, where S is the capital to land ratio. Note that h_s is housing supplied per unit of land area and is not the same as h in the household problem.

   The developer's problem be given by

   $$\max_{S} pS^{2/3} - iS - R$$

   where p, i and R are the costs of housing, capital, and land, respectively. For the remainder of the problem, let $i = \frac{1}{33}$.

   Solve for ${h_s}^*$ in terms of p.

   (e) (5) How does building height vary with distance to the center?

2. Consider the monocentric city model without housing. Assume we have a linear, open city. Let w=3, $\bar{l} = 1$, p_c = 1, $\overline{R}$ = 0.5, and $\overline{u} = 0$. Let $u(c) = \ln(c-1)$. Assume we are in a spatial equilibrium, so everyone is optimizing and no one wants to move.

(a) (10) Find c^*.

(b) (10) Find an expression for the edge of the city, $\overline{x}$, in terms of $w, c^*, \overline{R}, \bar{l}$ and t.

(c) (10) Use the assumption that there is one unit of land at each x to derive an expression for N^* in terms of $\overline{x}$ and $\bar{l}$.

(d) (10) Use the household's equilibrium budget constraint and the equilibrium extent of the city to solve for the equilibrium rent gradient, $R^*(x)$.

3. Consider the monocentric city model without housing described by the following notation.

$$\begin{aligned} c &\sim \text{consumption} \\ u(A(x)c) &\sim \text{utility} \\ \overline{u} &\sim \text{reservation utility} \\ x &\sim \text{location} \\ w &\sim \text{wage} \\ \bar{l} &\sim \text{land consumption} \\ R(x) &\sim \text{land rent} \\ \overline{R} &\sim \text{reservation land rent} \end{aligned}$$

This is the basic set up for the monocentric city model, with one change. Amenities vary over space to reflect changes in school quality. In particular, for $x < x^*, A(x) = A_0$ and for $x \geq x^*, A(x) = A_1$ where $A_0 < A_1$. Assume that $x^* < \overline{x}$ so that the change in school quality happens inside the limits of the city.

(a) (10) By how much does equilibrium consumption change when we move from just inside x^* to just outside?

(b) (10) By how much does equilibrium land rent change when we move from just inside x^* to just outside?

(c) (5) Draw a picture, with x on the horizontal axis and wages and prices on the vertical axis, that shows the equilibrium land rent gradient, wages, and consumption for this city.

(d) (5) In Black's (1999) paper, she estimates that when crossing the street from one school attendance zone to another, if attendance zone test scores increase by 1 point, then house prices will increase by 1-3%. Your answers to the two questions above suggest that thes statistics can be given a precise welfare interpretation. What is it?

1a HOUSEHOLD SOLVES

$$\max \ c^{1/2} h^{1/2} \qquad (1)$$

$$\text{S.T.} \quad \tilde{\omega} = ph + c \qquad (2)$$

$$(2) \Rightarrow \quad c = \tilde{\omega} - ph \qquad (3)$$

$$(3) \to (1) \Rightarrow \quad \max \ (\tilde{\omega} - ph)^{1/2} h^{1/2} \qquad (4)$$

$$\frac{d}{dh}(\cdot) = 0 \Rightarrow$$

$$\frac{1}{2}(\tilde{\omega} - ph)^{-1/2}(-p) h^{1/2} + (\tilde{\omega} - ph)^{1/2} \frac{1}{2} h^{-1/2} = 0$$

$$\Rightarrow \quad -ph + (\tilde{\omega} - ph) = 0$$

$$\Rightarrow \quad 2ph = \tilde{\omega} \Rightarrow h^* = \frac{\tilde{\omega}}{2p} \qquad (5)$$

b. NOW FIND p^* IN TERMS OF $\tilde{\omega}$. WE NEED TO USE THE FREE MOBILITY CONDITION, $u(c^*, h^*) = 3$. (6)

SUBSTITUTING (5) $\to$ (4) AND USING (6) GIVES

$$\left(\tilde{\omega} - p\left(\frac{\tilde{\omega}}{2p}\right)\right)^{1/2} \left(\frac{\tilde{\omega}}{2p}\right)^{1/2} = 3$$

WE JUST NEED TO SOLVE FOR P.

$$\Rightarrow \quad \left(\frac{\tilde{\omega}}{2}\right)^{1/2} \left(\frac{\tilde{\omega}}{2p}\right)^{1/2} = 3$$

$$\Rightarrow \quad \frac{\tilde{\omega}}{2} = 3\sqrt{p}$$

$$\Rightarrow \quad p^* = \left(\frac{\tilde{w}}{6}\right)^2 = \frac{\tilde{w}^2}{36} \quad (7)$$

c. TO GET h^* IN TERMS OF $\tilde{w}$

SUBSTITUTE (7) → (5)

$$\Rightarrow \quad h^* = \frac{\tilde{w}}{2\left(\frac{\tilde{w}^2}{36}\right)}$$

$$\Rightarrow \quad h^* = \frac{18}{\tilde{w}} = \frac{18}{w - 2tx}$$

d. DEVELOPER SOLVES

$$\text{MAX } p\, s^{2/3} - i s - T \qquad , i = 1/33$$

$$\frac{d}{ds}(\cdot) = 0 \Rightarrow \frac{2}{3} p s^{-1/3} - i = 0$$

$$\Rightarrow \quad s^{-1/3} = \frac{3i}{2p}$$

$$\Rightarrow \quad \frac{2p}{3i} = s^{1/3}$$

$$\Rightarrow \quad s^* = \left(\frac{2p}{3i}\right)^3$$

WITH $i = 1/3$ $\Rightarrow$ $s^* = \left(\frac{2p}{(3/33)}\right)^3 = (22p)^3$

$$h_s^* = (s^*)^{2/3} = \left[(22p)^3\right]^{2/3} = (22p)^2 \quad (8)$$

$$= 484p^2$$

(c) FROM (7) P^* IS DECREASING IN $\tilde{\omega}$. FROM THE DEFINITION OF $\tilde{\omega}$, $\tilde{\omega}$ IS DECREASING IN X. THUS $P^* \downarrow$ AS $X \uparrow$. FROM (8) h_s IS INCREASING IN P^*. PUTTING THIS ALL TOGETHER, $h_s \downarrow$ AS $x \uparrow$.

2. $\omega = 3$, $\bar{l} = 1$, $P_c = 1$, $\bar{R} = 1/2$, $\bar{u} = 0$

$u(c) = \ln(c-1)$

(a) FIND c^* SUCH THAT $u(c) = \bar{u}$

$$\Rightarrow \ln(c^* - 1) = 0 \Rightarrow c^* = 2$$

(b) HOUSEHOLD SOLVES

$$\max \; u(c)$$
$$\text{S.T.} \quad \omega = c + 2tx + R\bar{l}$$

$$\Rightarrow \quad \omega - c^* = 2tx + R\bar{l} \qquad (1)$$

AT $\bar{x}$ $\Rightarrow$ $\omega - c^* = 2t\bar{x} + \bar{R}\,l$

USING GIVEN VALUES AND $c^* = 1$

$$\Rightarrow 3 - 2 = 2t\bar{x} + \frac{1}{2} \cdot 1$$

$$\Rightarrow \frac{1}{2} = 2t\bar{x} \Rightarrow \bar{x} = \frac{1}{4t}$$

(c) EACH HOUSEHOLD CONSUMES $\bar{l}$ OF LAND

SO $N = \dfrac{2\bar{x}}{\bar{l}} = \dfrac{2\left[\frac{1}{4t}\right]}{1} = \dfrac{1}{2t}$

(d) FROM ①, AT ALL <u>OCCUPIED</u> LOCATIONS, WE HAVE

$$w - c^* = 2tx + R(x)\,\bar{l}$$

$$\Rightarrow \quad R(x) = \frac{w - c^* - 2t|x|}{\bar{l}} = \frac{3 - 2 - 2tx}{1}$$

AND

$$R(x) = \bar{R} \quad \text{AT UNOCCUPIED LOCATIONS.}$$

USING THE DATA, THIS GIVES

$$R(x) = \begin{cases} 1 - 2t|x| & |x| < \frac{1}{4t} \\ \frac{1}{2} & > \end{cases}$$

(3) (a) AMENITIES ARE

$$A(x) = \begin{cases} A_0 & x < x^* \\ A_1 & x > x^* \end{cases}$$

SO $\quad u(A(x)c) = \bar{u}$

$$\Rightarrow \quad A(x)c = u^{-1}(\bar{u})$$

$$\Rightarrow \quad c^*(x) = \frac{u^{-1}(\bar{u})}{A(x)}$$

OR

$$c^*(x) = \begin{cases} \dfrac{u^{-1}(\bar{u})}{A_0} & x < x^* \\ \dfrac{u^{-1}(\bar{u})}{A_1} & x > x^* \end{cases} \qquad (1)$$

WITH $A_1 > A_0$ THIS MEANS CONSUMPTION IS SMALLER FOR $x > x^*$ THAN FOR $x < x^*$.

SO WHEN WE CROSS FROM $x^* - \varepsilon$ TO $x^* + \varepsilon$ THE CHANGE IN CONSUMPTION IS

$$\Delta c = u^{-1}(\bar{u}) \left[\frac{1}{A_0} - \frac{1}{A_1} \right] > 0 \qquad (2)$$

(b) WE KNOW THAT FOR OCCUPIED LOCATIONS,

$$R(x) = \frac{w - c^* - 2t|x|}{\bar{l}}$$

WE JUST NEED TO SUBSTITUTE IN $c(x)$ FROM (1). SO

$$R(x) = \frac{w - c^*(x) - 2t|x|}{\bar{l}}$$

WE CAN USE THIS TO EVALUATE $\Delta R = R(x - \varepsilon) - R(x + \varepsilon)$ FOR ε SMALL

$$\Delta R = \left[\frac{w - c^*(x^* - \varepsilon) - 2t|x^* - \varepsilon|}{\bar{l}}\right] - \left[\frac{w - c^*(x^* + \varepsilon) - 2t|x^* + \varepsilon|}{\bar{l}}\right]$$

$$= \frac{1}{\bar{l}}\left[c^*(x^* + \varepsilon) - c^*(x - \varepsilon) - 2t(2\varepsilon)\right]$$

FOR $\varepsilon \to 0$ WE CAN IGNORE THE TERM INVOLVING ε

$$\Rightarrow \quad \Delta R = \frac{1}{\bar{l}}\left[c^*(x^* + \varepsilon) - c^*(x - \varepsilon)\right]$$

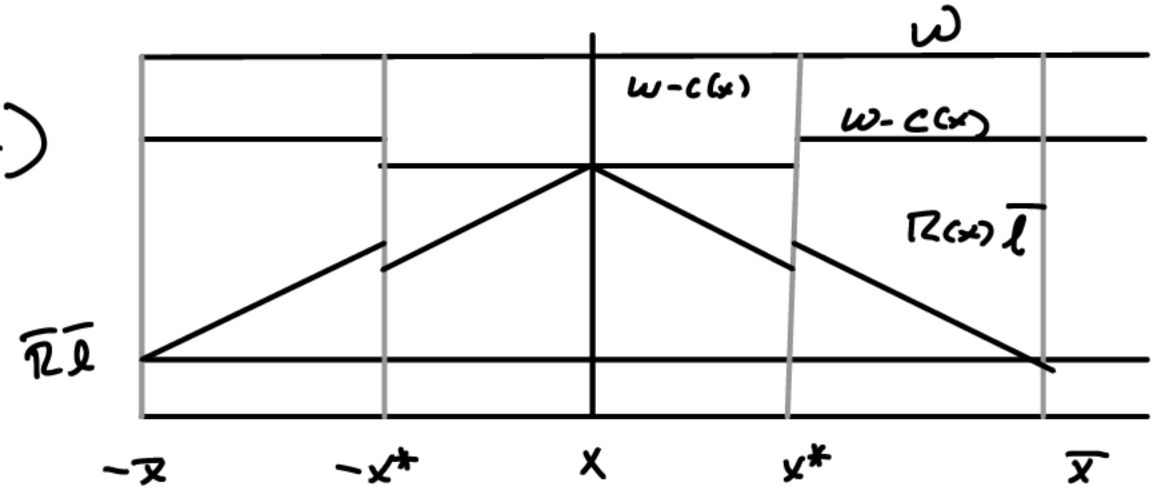
USING (2) THIS GIVES

$$\Delta R = \frac{1}{\bar{l}}\left[-\Delta c\right]$$

$$= -\frac{u^{-1}(\bar{u})}{\bar{l}}\left[\frac{1}{A_0} - \frac{1}{A_1}\right] < 0$$

SO CHANGES IN RENT AND CONSUMPTION EXACTLY OFFSET.

(c)



(d) WHEN WE MOVE OUT FROM THE CENTER, ACROSS THE SCHOOL DISTRICT BOUNDARY FROM THE BAD SCHOOL DISTRICT TO THE GOOD SCHOOL DISTRICT, CONSUMPTION MUST FALL SO THAT ALL HOUSEHOLDS HAVE THE SAME UTILITY. THIS CAN ONLY HAPPEN IF RENTS RISE.

THUS, THE INCREASE IN RENT AT THE SCHOOL DISTRICT BOUNDARY IS EXACTLY ENOUGH THAT THE CORRESPONDING CHANGE IN CONSUMPTION HAS THE SAME VALUE TO HOUSEHOLDS AS THE CHANGE IN SCHOOL QUALITY.