EC1410 – Spring 2023

Midterm
8:30-9:30am, April 5, 2023
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You will have 60 minutes to complete this exam. No notes or books are allowed, but you may use a calculator. Cell phones and any device with a wireless connection must be off. Anyone still working on their exam after time is called will be subject to an automatic 10 point penalty.

When you write up your answers, your goal should be to (1) be correct, and (2) convince your reader that your answer is correct. Answers which do not achieve these goals will not be awarded full credit. To accomplish the second goal, it is helpful if your work is legible and if all steps are presented, possibly with a line of explanation. Total points =100. Points assigned to each problem are indicated in parentheses.

This exam has TWO pages.

1. In this problem, we will work through an example of the monocentric city model. Assume we have a linear, open city. Let w=4, $\bar{l} = 1$, p_c = 1, $\overline{R}$ = 0.5, $\overline{u} = \sqrt{2}$, and $u(c) = \sqrt{c}$.

   (a) (10) State the two conditions that must be satisfied by a spatial equilibrium in this monocentric city model.

   (b) (10) Find c^*, the equilibrium level of consumption for all households.

   (c) (10) Using the constraint from the household's problem, find an expression for the edge of the city, $\overline{x}$, in terms of $w, c^*, \overline{R}, \bar{l}$ and t.

   (d) (10) Use the assumption that there is one unit of land at each x to derive an expression for city population N^* in terms of $\overline{x}$ and land consumption, $\bar{l}$.

   (e) (10) Use the household's equilibrium budget constraint and the equilibrium extent of the city to solve for the equilibrium rent gradient, $R^*(x)$.

2. Consider the monocentric city model with housing. Let $\overline{u} = 3$ and suppose that the household problem is

$$\max_{c,h,x} c^{1/2}h^{1/2} \text{ subject to } w = c + ph + 2tx$$

   (a) (5) Let $\widetilde{w} = w - 2tx$. Use the first-order condition of the household's problem with respect to h to show that $h^* = \widetilde{w}/2p$.

   (b) (5) Use the fact that utility is $\overline{u} = 3$ everywhere and the results from (a) to solve for p^* in terms of $\widetilde{w}$.

   (c) (5) Substitute your expressions for p^* and $\widetilde{w}$ into your expression for h^* to write h^* in terms of w, t, and x.

(d) (5) Suppose that S is the capital to land ratio, and that the amount of housing supplied at location x is $h_S = S^{2/3}$. Let p, $i = \frac{1}{33}$ and R be the costs of housing, capital, and land, respectively. The real estate developers problem is

$$\max_S pS^{2/3} - iS - R$$

Use the first-order condition of this problem with respect to S to solve for $h_S{}^*$ in terms of p.

(e) (5) Does this model require that building heights decline with distance to the center? Explain briefly.

3. In this problem, we consider the monocentric city model without housing, but two types of agents. The two types of agents are automobile drivers (a) and bus riders (b), with $w_a = 10$ and $w_b = 5$. Assume that transportation costs for the two types are

$$(wt^b + c^b)|x| \text{ for bus riders } t^b = 2, c^b = 1 > 0$$
$$f + (wt^a + c^a)|x| \text{ for drivers, where } t^a = 1, c^a = 2$$

In addition, assume $u(c) = \ln(c-1)$, $\overline{u} = 0$, $\overline{R} = 0$, $\overline{\ell} = 1$ and $p_c = 1$.

(a) (5) Set up the household problem for each type of agent.

(b) (5) Assume $u(c^*) = \overline{u}$ for both types. Find c^*.

(c) (5) Find the bid rent functions $R_a(x)$ and $R_b(x)$, for both types.

(d) (10) Suppose $f = 5$ and plot $R_a(x)$ and $R_b(x)$ on one graph. Indicate the areas in which each type has the higher willingness to pay. Describe the resulting equilibrium briefly.

1. CONSIDER A LINEAR, MONOCENTRIC, OPEN CITY WITH $w=4$, $\bar{\ell}=1$, $P_c=1$, $\bar{R}=\frac{1}{2}$, $\bar{u}=\sqrt{2}$, $u(c)=\sqrt{c}$

(a) IN EQUILIBRIUM, WE REQUIRE THAT HOUSEHOLDS OPTIMIZE

$$\max\ u(c)$$
$$\text{s.t.}\quad w = c + 2tx + R\bar{\ell} \qquad (1)$$

AND NO ONE WANTS TO MOVE / FREE MOBILITY

$$u(c)=\bar{u} \quad \text{ALL } x.$$

(b) WITH $\bar{u}=\sqrt{2}$ AND $u(c)=\sqrt{c}$

FREE MOBILITY REQUIRES THAT

$$\sqrt{c^*}=\sqrt{2} \quad \text{ALL } x$$
$$\Rightarrow \quad c^*=2$$

(c) AT THE EDGE OF THE CITY $R(x)=\bar{R}$, SO (1) BECOMES

$$w=c+2t\bar{x}+\bar{R}\bar{\ell}$$

SUBSTITUTING GIVEN VALUES FOR $w, \bar{R}, \bar{\ell}$ AND c^* FROM PART (b), WE HAVE

$$4=2+2t\bar{x}+\frac{1}{2}\cdot 1$$
$$\Rightarrow \quad \bar{x}=\frac{3}{2}\cdot\frac{1}{2t}=\frac{3}{4t}$$

(d) WITH $\bar{x} = \frac{3}{4t}$ AND $\bar{\ell} = 1$,

$$N^* = \frac{2\bar{x}}{\bar{\ell}} = \frac{3}{2t}$$

(e) USING THE HOUSEHOLD BUDGET CONSTRAINT,

$$w = c^* + 2tx + R\bar{\ell}$$

WE HAVE (FOR $|x| < \bar{x}$)

$$4 = 2 + 2tx + R \cdot 1$$

$$\Rightarrow R(x) = 2 - 2t|x|$$

SO $R(x) = \begin{cases} 2 - 2t|x| & |x| < \frac{3}{4t} \\ \frac{1}{2} & > \end{cases}$

2. $\bar{u} = 3$ AND HOUSEHOLD SOLVES

$$\max\ c^{1/2} h^{1/2}$$

$$\text{s.t. } w = c + ph + 2tx$$

(a) WITH $\tilde{w} = w - 2tx$ WE HAVE

$$\max\ c^{1/2} h^{1/2}$$

$$\text{s.t. } \tilde{w} = c + ph$$

SOLVING CONSTRAINT FOR c, WE HAVE

$$c = \tilde{w} - ph$$

SUBSTITUTING INTO $u(c,h)$, H.H. PROBLEM

IS
$$\max \ (\tilde{\omega}-ph)^{1/2}(h)^{1/2} \qquad (**)$$

F.O.C. IS

$$\frac{1}{2}(\tilde{\omega}-ph)^{-1/2}(-p)h^{1/2} + (\tilde{\omega}-ph)^{1/2}\frac{1}{2}h^{-1/2} = 0$$

$$\Rightarrow \quad \frac{1}{2}(-p)h^{1/2} + (\tilde{\omega}-ph)\frac{1}{2}h^{-1/2} = 0$$

$$\Rightarrow \quad \frac{-p}{2}h + (\tilde{\omega}-ph)\frac{1}{2} = 0$$

$$\Rightarrow \quad -ph + \tilde{\omega}-ph = 0$$

$$\Rightarrow \quad 2ph = \tilde{\omega} \Rightarrow h^* = \frac{\tilde{\omega}}{2p} \qquad (*)$$

(b) USING (*) IN (**), WE HAVE

$$\left(\tilde{\omega} - p\frac{\tilde{\omega}}{2p}\right)^{1/2}\left(\frac{\tilde{\omega}}{2p}\right)^{1/2} = 3$$

$$\Rightarrow \quad \left(\frac{\tilde{\omega}}{2}\right)^{1/2}\left(\frac{\tilde{\omega}}{2p}\right)^{1/2} = 3$$

$$\Rightarrow \quad \frac{\tilde{\omega}}{2}\cdot\frac{\tilde{\omega}}{2p} = 9$$

$$\Rightarrow \quad p^* = \frac{\tilde{\omega}^2}{36} \qquad (***)$$

(c) USING (****) IN (*) WE HAVE

$$h^* = \frac{\tilde{w}}{2p^*}$$

$$= \frac{\tilde{w}}{2 \cdot \left(\frac{\tilde{w}^2}{36}\right)}$$

$$= \frac{18}{\tilde{w}} = \frac{18}{w - 2tx}$$

(d) DEVELOPER PROBLEM IS

$$\max_{s} \; p s^{2/3} - i s - R$$

F.O.C. $\quad p \frac{2}{3} s^{-1/3} - i = 0$

$$\Rightarrow \quad s^{-1/3} = \frac{3i}{2p}$$

$$\Rightarrow \quad s^* = \left(\frac{2p}{3i}\right)^3 = (2zp)^3$$

RECALLING THAT $h = s^{2/3}$

$$h_s^* = \left(\frac{2p}{3i}\right)^2 = (2zp)^2$$

(e) BUILDING HIEGHT IS REPRESENTED IN THIS MODEL BY THE CAPITAL TO LAND RATIO $S^* = \left(\frac{2p}{3i}\right)^3$. FROM PART (b);

p^* IS DECREASING IN X. IT FOLLOWS THAT S^* MUST ALSO DECREASE IN X. THAT IS, BUILDING HIEGHT DECLINES WITH DISTANCE FROM THE CENTER.

3. (a) BUS RIDER PROBLEM

$$\max \ln(c-1)$$
$$\text{s.t. } w_b = c + R\bar{l} + (w_b t^b + c^b)x \quad (1)$$

CAR DRIVER PROBLEM

$$\max \ln(c-1)$$
$$\text{s.t. } w_a = c + R\bar{l} + f + (w_a t^a + c^a)x \quad (2)$$

(b) WITH $\bar{u}=0$ WE HAVE $\ln(c^*-1)=0$

$$\Rightarrow c^*-1 = e^0$$
$$\Rightarrow c^*-1 = 1$$
$$\Rightarrow c^*=2$$

(c) USING $c^*=2$ AND $t^b=2$, $c^b=1$, $w_b=5$, AND $\bar{l}=1$ IN (1)

$$5 = 2 + R_b + (5\cdot 2 + 1)x$$

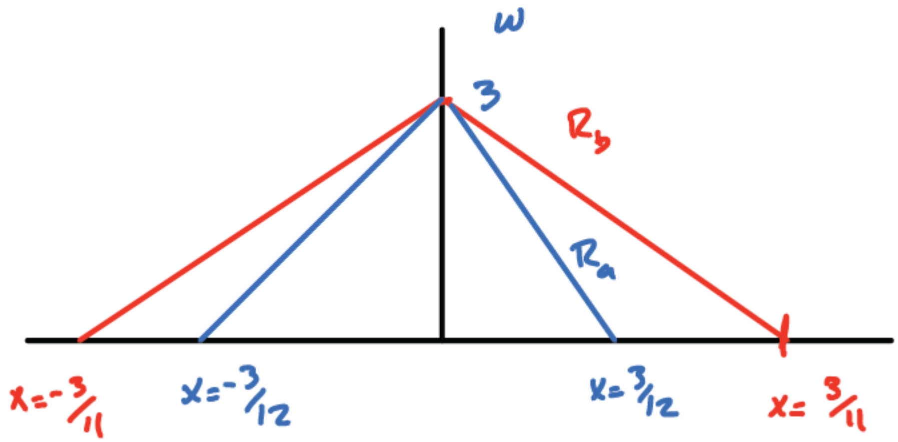
$$\Rightarrow R_b = 3 - 11x$$

SIMILARLY, USING $c^{*}=2, \bar{l}=1, t^{a}=1, c^{a}=2, w_{a}=10$ IN ②

$$10 = 2 + R_a + f + (10+2)x$$

$$\Rightarrow \quad R_a = 8 - f - 12x$$

(d) IF $f=5$ WE HAVE $R_a = 3 - 12x$



IN THIS CASE R_a IS EVERYWHERE LESS THAN R_b EXCEPT AT $x=0$, WHERE THEY ARE EQUAL.

$\Rightarrow$ BUS DRIVERS OUTBID CAR DRIVERS TO LIVE IN THE CITY AT ALL x, EXCEPT $x=0$, WHERE THE MODEL IS INDETERMINANT.